

# DIAGRAM GENUS, GENERATORS AND APPLICATIONS

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**Abstract.** We continue the study of the genus of knot diagrams, deriving a new description of generators using Hirasawa's algorithm. This description leads to good estimates on the maximal number of crossings of generators and allows us to complete their classification for knots of genus 4.

As applications of the genus 4 classification, we establish non-triviality of the skein polynomial on  $k$ -almost positive knots for  $k \leq 4$ , and of the Jones polynomial for  $k \leq 3$ . For  $k \leq 4$ , we classify the occurring achiral knots, and prove a trivializability result for  $k$ -almost positive unknot diagrams. This yields also estimates on the number of unknotting Reidemeister moves. We describe the positive knots of signature (up to) 4.

Using a study of the skein polynomial, we prove the exactness of the Morton-Williams-Franks braid index inequality and the existence of a minimal string Bennequin surface for alternating knots up to genus 4. We also prove for such knots conjectures of Hoste and Fox about the roots and coefficients of the Alexander polynomial.

*Keywords:* almost positive knot, genus, Jones polynomial, Alexander polynomial, skein polynomial, achiral knot, unknot diagram, braid index, Bennequin surface, signature

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## 1 Introduction

Introducing the *genus*  $g(K)$  of a knot  $K$ , Seifert [Se] gave a construction of compact oriented surfaces in 3-space bounding the knot (*Seifert surface*) by an algorithm starting with some diagram of the knot (see [Ad, §4.3] or [Ro]). The surface given by this algorithm is called *canonical*. A natural problem is when the diagram is *genus-minimizing*, or *of minimal genus*, that is, its canonical Seifert surface has minimal genus among all Seifert surfaces of the knot. This problem has been studied over a long period. First, the minimal genus property was shown for alternating diagrams, independently by Crowell [Cw] and Murasugi [Mu2]. Their proof is algebraical, using the Alexander polynomial  $\Delta$  [Al] and the inequality  $\max \deg \Delta \leq g$  (which thus they prove to be exact for alternating knots). Later Gabai [Ga] developed a geometric method using foliations and showed that this method is likewise successful for alternating diagrams. Murasugi [Mu3] introduced the operation  $*$ -product. It was shown to behave naturally w.r.t. the Alexander polynomial by himself, and later by Gabai [Ga2, Ga3] in the geometrical context. These results imply the extension of the minimal genus property of alternating diagrams to the homogeneous diagrams of [Cr]. An important other subclass of the class of homogeneous diagrams and links are the positive diagrams and links. Such links have been considered (in general or in special cases) independently before. (See e.g. [CG, CM, FW, O, Ru, Tr, Yo, Zu].) The minimal genus property for such diagrams follows from yet a different source, the work of Bennequin [Be] on contact structures. His inequality (theorem 3 in that paper; stated as theorem 2.3 below) in fact allows to estimate the difference between the genus of the diagram and the genus of the knot in terms of the number of positive or negative crossings.

The prospect of applications led to the treatment of canonical surfaces, and the establishment of the *canonical genus*, the minimal genus of all such surfaces for a given knot or link, in its own right. In this paper, we will continue the study of canonical Seifert surfaces from the combinatorial point of view. This study was initiated

myself in [St4], and independently by Brittenham [Br], and set forth in [St2], and later in [STV, SV]. (Some explanation of this work is given also in section 5.3 of Cromwell’s recent book [Cr2].) Using the theoretical insight gained there we had methods efficient enough to complete the classification up to genus 3, in terms of the list of “generators”. In [SV] the relation to certain algebraic objects named Wicks forms was discussed, and applied to the enumeration of alternating knots by genus. We gave an extensive list of other applications in [St2], and later for example in [St18, St12].

The previous method of [SV] does not apply well for several components, and thus we develop an alternative approach, which bases on the special diagram algorithm first found by Hirasawa [Hi2] (and rediscovered a little later independently in [St9, §7]). We will work out inequalities for the crossing number and number of  $\sim$ -equivalence classes of  $\tilde{\mathbb{Z}}_2$ -irreducible link diagrams (generators). So far we carried this out only for knots. The present approach allows to improve what we know in the knot case and extend it to links.

Then we treat the description, and applications, of the compilation of knot generators of genus 4. As has already become apparent in the preceding generator compilations for genus 2 and 3, and then also from the result of [SV], the growth of the number of generators is enormous. In order to push the task back within the limits of (reasonable) computability, we need important new theoretical knowledge. It relies heavily on the special diagram algorithm of Hirasawa and the previous careful analysis of this procedure we carry out.

The first application addresses the identification of the unknot. This is a basic problem in knot theory. There are general methods using braid foliations [BM2], and Haken theory [HL]. However, these methods are difficult to use in practice. So one is interested in applicable criteria, at least for special classes of diagrams.

It is known, for example from Crowell [Cw] and Murasugi [Mu2], that in alternating unknot diagrams all crossings are nugatory. For positive, as for alternating, diagrams, one can observe the same phenomenon using the Alexander polynomial (as in [Cr]). It is also an application of the Bennequin inequality (theorem 2.3 below), in its extended form based on the Vogel algorithm [Vo] (see [St]). For almost alternating diagrams a trivializability result using flypes and “tongue moves” (see figure 7) was proved recently by Tsukamoto [Ts], confirming a previous conjecture of Adams. Here we prove several related results, among others the following:

**Theorem 1.1** For  $k \leq 4$ , all  $k$ -almost positive unknot diagrams are trivializable by crossing number reducing wave moves and factor slides.

A wave move is shown in figure 6, and a factor slide in figure 2 (b).

For  $k$ -almost positive diagrams the description in the case  $k = 1$  (unknotted twist knot diagrams and possible nugatory crossings) was written down in [St3]. It was, though, previously known to Przytycki, who observed it as a consequence of Taniyama’s result [Ta]. Their joint work had been announced long ago, but the full paper was not finished until very recently [PT]. This description includes and concretizes Hirasawa’s result [Hi] on almost special alternating diagrams. For 2-almost positive diagrams (in particular 2-almost special alternating diagrams) the result is given in [St2]. Now we settle the cases  $k \leq 4$  in the stated way.

This theorem is also quite analogous to the result<sup>1</sup> in [NO] for 3-bridge diagrams, and for arborescent (or Conway-algebraic) diagrams, which is a consequence of Bonahon-Siebenmann’s classification of arborescent links. A partial writeup of their (still unpublished) work, which covers the treatment of the unknot, is given in [FG].

A tongue move is a special type of wave move, preserving the property ‘almost alternating’. From this point of view, we can extend Tsukamoto’s (and Hirasawa’s) result for  $k = 1$  in special diagrams to  $k \leq 5$  (see corollary 5.2 and theorem 5.1). In fact, this extension was one of the motivations for our attention to this type of problem. We can apply our work also to give polynomial estimates on the number of Reidemeister moves needed for unknotting, which has been recently studied by various authors [HL, HN, Hy] (see proposition 5.4).

From a different perspective, a factor slide is a (very special) type of preserving wave move. Then theorem 1.1 confirms in our case, in a stronger form, a conjecture (conjecture 5.1), stating that preserving and reducing wave moves in combination suffice to trivialize unknot diagrams. Example 5.1 shows that our stronger statement is not true in general, and the assumption  $k \leq 4$  in the theorem cannot be improved.

The work on theorem 1.1 allows us to address two related problems which have been given some attention in the literature. The following result relates to the non-triviality problem of link polynomials.

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<sup>1</sup>I was informed, though, of a possible gap in its proof.

**Theorem 1.2** Any non-trivial  $k$ -almost positive knot has non-trivial skein polynomial  $P$  for  $k \leq 4$  and non-trivial Jones polynomial  $V$  for  $k \leq 3$ .

So far, only the cases  $k = 0, 1$  were known (see [St6]). For  $P$ , the result follows from the proof of theorem 1.1 almost directly. For  $V$ , we combine the skein and Kauffman bracket properties of the Jones polynomial in [BMo, St6, St8] with some arguments about the signature and Gabai’s geometric work [Ga2, Ga3]. (They allow us to reduce the necessary computations to a minimum.)

The proof of theorem 1.1 can be applied in some form not only to the trivial, but more generally to an amphicheiral (achiral) knot (see corollary 5.4). We can then obtain the classification of these knots.

**Theorem 1.3** The  $k$ -almost positive achiral knots for  $k \leq 4$  are those of  $2k$  crossings, that is,  $4_1$  for  $k = 2$ ;  $6_3$  and  $3_1\#1_3$  for  $k = 3$ ; and  $8_3, 8_9, 8_{12}, 8_{17}, 8_{18}$ , and  $4_1\#4_1$  for  $k = 4$ .

This property was known for  $k \leq 2$  by [PT]; prior to latter’s completion, written account was given in [St6] for  $k \leq 1$  and [St2] for  $k = 2$ . Theorem 1.3 adds the cases  $k = 3, 4$ . It was obtained in [St6] for alternating prime knots (independently provable using the work in [Th]), and checked there also for prime knots of  $\leq 16$  crossings. The proof of theorem 1.3 will consist in extending the arguments for theorem 1.1 and simplifying diagrams of such knots to these checked low crossing cases (see corollary 5.4).

A final application concerning positivity is the determination of the positive knots of signature 4 (theorem 6.1). Again the result for signature 2 (namely, that these are precisely the positive knots of genus 1) is a consequence of Taniyama’s work [Ta]. Our theorem is the first non-trivial explicit step beyond Taniyama toward the general case of the conjecture that positive knots of only finitely many genera have given signature; see [St15, St12].

The later sections of the paper extend the applications of generators and regularization to alternating knots. In §7, we use a regularization of the skein polynomial and the work of Murasugi-Przytycki [MP] to show

**Theorem 1.4** The Morton-Williams-Franks braid index inequality MWF is exact on alternating knots of genus up to 4.

We will also confirm a conjecture of Murasugi-Przytycki (conjecture 2.1) for such knots. Theorem 1.4 complements the results in [Mu] for fibered and 2-bridge knots; it comes close to the examples of strict MWF inequality of a 4-component genus 3 (alternating) link and a genus 6 knot found in [MP]. Evidence from some computations during its proof led us to conjecture an improvement of Ohya’s [Oh] braid index inequality for special alternating links (§7.5). We confirm this stronger inequality (apart from knots of genus up to 4) also for arborescently alternating links.

Our study of Murasugi-Przytycki’s index requires also to clarify and correct an inexactness in [MP], which affects the proof of their main upper braid index estimate (see §7.2). We revealed a slight discrepancy between their definition of graph index and the diagrammatic move they introduce to reduce the number of Seifert circles. We will be able to justify their estimates, but still there is some cost, in that the correspondence between a diagram and (its Seifert) graph becomes (in general) lost during the recursive calculation of the index (using their old definition). This oversight seems to propagate to other papers and may cause a problem at some point. We will explain how to define, on the level of Seifert graphs, the “right” index w.r.t. their move.

In §8 we develop a method of constructing a Bennequin surface of links for a minimal genus canonical surface. The number of strings of the resulting braid is generally low, and can be calculated by modifying our corrected version of graph index. We apply this construction to alternating knots of genus up to 4, and show that one can span for these knots a Bennequin surface on a minimal string braid. Bennequin [Be] proved such a result for all 3-braid links, and Hirasawa (unpublished) for the 2-bridge links. In contrast, knots lacking such surfaces are known for braid index 4 and genus 3. We will see how to apply the Murasugi-Przytycki move keeping a minimal genus surface, finally converting it into a minimal string Bennequin surface.

The final section §9 deals with the Alexander polynomial of an alternating knot. We consider first a conjecture of Hoste about the roots of the Alexander polynomial. We develop several tests based on the generator description, and apply them to confirm this conjecture on knots up to genus 4 (theorem 9.1).

Another conjecture we treat is the log-concavity conjecture of [St16]. It states that the square of each coefficient of the Alexander polynomial of an alternating link is not less than the product of its two neighbors. This implies Fox’s “trapezoidal” conjecture 9.3. We will verify both conjectures for knots up to genus 4 (theorem 9.5). In this case it was possible to extend the result to links (theorem 9.6).

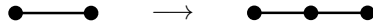
As another qualitative extension of the Fox conjecture, we will conclude that the convex hull of alternating knot polynomials of given genus is a polytope, and we can determine this polytope (i.e. the complete set of linear inequalities satisfied by the coefficients of Alexander polynomials of alternating knots) up to genus 4 in §9.3. Apart from an improvement of the trapezoidal inequalities, we will compare our result also to the inequalities obtained by Ozsváth-Szabó using knot Floer homology [OS]. (Their inequalities give an alternative proof of the trapezoidal conjecture for genus 2.) In-Dae Jong [Jn] used the generator description (theorem 2.12) to prove the log-concavity conjecture and then, following the discussion in §9.3, to give the complete set of linear inequalities for genus 2.

## 2 Preliminaries

### 2.1 Graphs

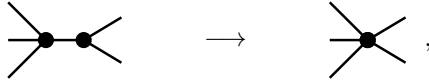
A graph  $G$  will have for us possibly multiple edges (edges connecting the same two vertices), but usually no loop edges (edges connecting one and the same vertex). By  $V(G)$  we will denote the set of vertices of  $G$ , and by  $E(G)$  the set of edges of  $G$  (each multiple edge counting as a set of single edges);  $v(G)$  and  $e(G)$  will be the number of vertices and edges of  $G$  (thus counted), respectively.

For a graph, let the operation



(adding a vertex of valence 2) be called *bisecting* and its inverse (removing such a vertex) *unbisecting* (of an edge). We call a graph  $G'$  a *bisection* of a graph  $G$  with no valence-2-vertices, if  $G'$  is obtained from  $G$  by a sequence of edge bisections. We call a bisection  $G'$  *reduced*, if it has no adjacent vertices of valence 2 (that is, each edge of  $G$  is bisected at most once). Contrarily, if  $G'$  is a graph, its *unbisected graph*  $G$  is the graph with no valence-2-vertices, of which  $G'$  is a bisection.

Similarly, a *contraction* is the operation



and a *decontraction* its inverse.

The *doubling* of an edge consists in adding a new edge connecting the same two vertices.

A graph is *n-connected*, if  $n$  is the minimal number of edges needed to remove from it to disconnect it. (Thus connected means 1-connected.) Such a collection of edges is called an *n-cut*.

Hereby, when we delete an edge, we understand that a vertex it is incident to is *not* to be deleted too. In that sense, the set  $I_v$  of edges incident to a given vertex  $v$  always forms a cut – if we delete these edges,  $v$  gets an isolated component, so the graph is not connected anymore. A *cut vertex* is a vertex which disconnects a graph, when removed *together* with all its incident edges.

A graph  $G$  is *planar* if it is embeddable in the plane *and equipped* with a fixed such planar embedding. Observe that there is a natural bijection of edges between a planar graph  $G$  and its dual graph  $G^*$ ; in that sense we can talk of the dual  $e^* \in E(G^*)$  of an edge  $e \in E(G)$ . The operations doubling and bisection become dual to each other.

## 2.2 Knots and diagrams

A crossing  $p$  in a knot diagram  $D$  is called *reducible* (or nugatory) if it looks like on the left of figure 1.  $D$  is called *reducible* if it has a reducible crossing, else it is called *reduced*. The reducing of the reducible crossing  $p$  is the move depicted on figure 1. Each diagram  $D$  can be (made) reduced by a finite number of these moves.

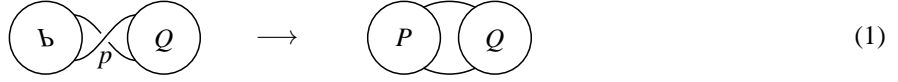


Figure 1

We assume in the following all diagrams reduced, unless otherwise stated.

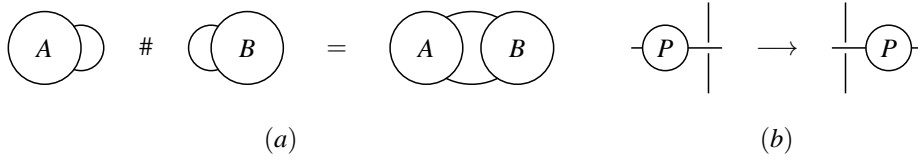


Figure 2: Diagram connected sum and factor slide move

Part (a) of figure 2 displays the *connected sum*  $D = A \# B$  of the diagrams  $A$  and  $B$ . Latter are called *factors* of  $D$ . The connected sum on diagrams is uniquely determined only up to the move shown in part (b) of the figure (and the mirror image of that move). We will call it a *factor slide*. If a diagram  $D$  can be represented as the connected sum of diagrams  $A$  and  $B$ , such that both  $A$  and  $B$  have at least one crossing, then  $D$  is called *composite*, otherwise it is called *prime*. A knot of link  $K$  is *prime* if whenever  $D = A \# B$  is a composite diagram of  $K$ , one of  $A$  and  $B$  represent an unknotted arc (but not both; the unknot is not prime per convention).

A diagram  $D$  is *connected* if its curve is a connected set in  $\mathbb{R}^2$ , that is, there is no closed curve  $\gamma$  disjoint from  $D$  such that both the interior and exterior of  $\gamma$  have non-trivial intersection with  $D$ . Otherwise  $D$  is *disconnected* or *split*. A link is *split* if it has a split diagram, and otherwise *non-split*.

**Theorem 2.1** ([Me]) If  $D$  has a prime alternating non-trivial diagram of  $K$ , then  $K$  is prime.

The (Seifert) *genus*  $g(K)$  resp. *Euler characteristic*  $\chi(K)$  of a knot or link  $K$  is said to be the minimal genus resp. maximal Euler characteristic of Seifert surface of  $K$ . For a diagram  $D$  of  $K$ ,  $g(D)$  is defined to be the genus of the Seifert surface obtained by Seifert's algorithm on  $D$ , and  $\chi(D)$  its Euler characteristic. Let  $c(D)$  denote the *number of crossings* of  $D$  and  $n(D) = n(K)$  the *number of components* of  $D$  or  $K$  (so  $n(K) = 1$  if  $K$  is a knot). Write  $s(D)$  for the *number of Seifert circles* of  $D$ . Then  $\chi(D) = s(D) - c(D)$  and  $2g(D) = 2 - n(D) - \chi(D)$ .

**Theorem 2.2** (see [Mu2, Cw, Ga, Cr]) If  $D$  has an alternating or positive diagram of  $K$ , then  $g(K) = g(D)$  and  $\chi(D) = \chi(K)$ .

The *crossing number*  $c(K)$  is the minimal crossing number of all diagrams  $D$  of  $K$ . The *canonical genus*  $\tilde{g}(K)$  resp. *canonical Euler characteristic*  $\tilde{\chi}(K)$  is defined as the minimal genus resp. maximal Euler characteristic of all diagrams of  $K$ . In general we can have  $g(K) < \tilde{g}(K)$ , that is, no diagrams of  $K$  of minimal genus (see [Mo]).

The *writhe*, or (*skein*) *sign*, is a number  $\pm 1$ , assigned to any crossing in a link diagram. A crossing as in figure 3(a) has writhe 1 and is called *positive*. A crossing as in figure 3(b) has writhe  $-1$  and is called *negative*. The *writhe* of a link diagram is the sum of writhes of all its crossings.

Let  $c_{\pm}(D)$  be the number of positive, respectively negative crossings of a diagram  $D$ , so that  $c(D) = c_{+}(D) + c_{-}(D)$  and  $w(D) = c_{+}(D) - c_{-}(D)$ . Let  $c_{\pm}(K)$  for a knot  $K$  denote the minimal number of positive resp. negative crossings of a diagram of  $K$ .

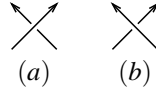


Figure 3

A diagram is *positive* if all its crossings are positive. A diagram is *almost positive* if all its crossings are positive except exactly one. A knot is *positive* if it has a positive diagram. (See e.g. [CM, O, Yo, Zu].) It is *almost positive* if it is not positive but has an almost positive diagram. More generally a diagram  $D$  is *k-almost positive* if it has exactly  $k$  negative crossings, i.e.  $c_-(D) = k$ , and a knot  $K$  is *k-almost positive* if it has a  $k$ -almost positive, but no  $k - 1$ -almost positive diagram, i.e.  $c_-(K) = k$ . We will sometimes call the switch of crossings of a diagram  $D$  so that all become positive the *positification* of  $D$ .

*Bennequin's inequality* (theorem 3 in [Be]) can be stated, using (as explained in [St]) the work by Vogel [Vo], thus:

**Theorem 2.3** If  $D$  is a diagram of a knot  $K$ , then  $g(K) \geq g(D) - c_-(D)$ .

In particular, if  $D$  is a diagram of the unknot, then  $c_-(D) \geq g(D)$ .

We call a crossing  $p$  connected to a Seifert circle  $s$  also *adjacent* or *attached* to  $s$ . The *valency* of a Seifert circle  $s$  is the number of crossings attached to  $s$ . We call a Seifert circle *negative* if only negative crossings are attached to it. Let  $s_-(D)$  be the number of negative Seifert circles of  $D$ .

Bennequin's inequality was improved by Rudolph [Ru].

**Theorem 2.4** If  $D$  is a diagram of a knot  $K$ , then

$$g(K) \geq g_s(K) \geq g(D) - c_-(D) + s_-(D). \quad (2)$$

Here  $g_s(K)$  is the *smooth slice genus* of  $K$ .

We will refer to (2) as the *Rudolph-Bennequin inequality*.

A diagram is *special* if no Seifert circle contains other Seifert circles in both regions it separates the plane into. Such Seifert circles are called *separating*. It is an easy observation that for connected diagrams two of the properties alternating, positive/negative and special imply the third. A diagram with these properties is called *special alternating*. A knot is special alternating if it has a special alternating diagram. Such knots were introduced and studied by Murasugi [Mu] and have a series of special features. Contrarily, all knots have a special (not necessarily alternating) diagram. Hirasawa [Hi2] shows how to modify any knot diagram  $D$  into a special diagram  $D'$  so that  $g(D) = g(D')$ . (Actually, the canonical surfaces of  $D$  and  $D'$  are isotopic.)

A diagram is *almost alternating* [Ad, Ad3, Ad2, GHY] if it can be turned by one crossing change into an alternating one. A knot is *almost alternating*, if it has an almost alternating diagram, but is not alternating.

The *index*  $\text{ind}(s)$  of a separating Seifert circle  $s$  is defined as follows: denote for an inner crossing (i.e., attached from the inside) of  $s$  a letter ' $i$ ', and for an outer crossing a letter ' $o$ ' cyclically along  $s$ . Then  $\text{ind}(s)$  is by definition the minimal number of disjoint subwords of the form  $i^n$  or  $o^n$  ( $n > 0$ ) of this cyclic word. For example, the Seifert circle  $s$  in figure 9 on page 17 has index 4.

## 2.3 Diagrammatic moves

**Definition 2.1** A *flype* is a move on a diagram shown in figure 4. We say that a crossing *admits* a flype if it can be represented as the distinguished crossing in the diagrams in the figure, and both tangles have at least one crossing.

By the fundamental work of Menasco-Thistlethwaite, we have a proof of the Tait flying conjecture.

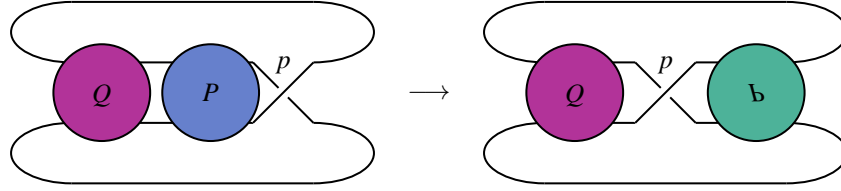
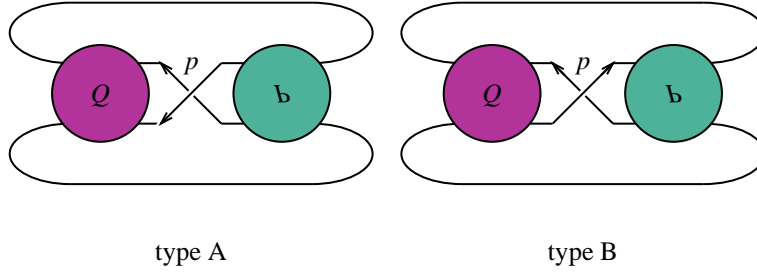
Figure 4: A flype near the crossing  $p$ 

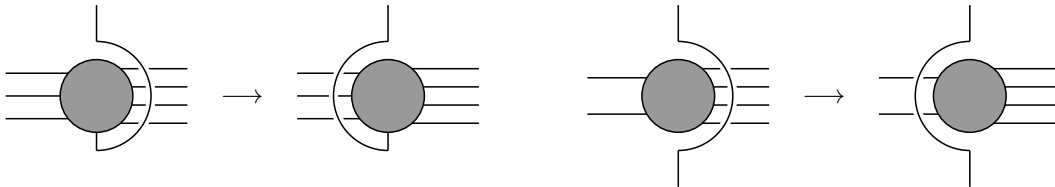
Figure 5: A flype of type A and B

**Theorem 2.5** ([MT]) For two alternating diagrams of the same prime alternating link, there is a sequence of flypes taking the one diagram into the other.

We introduced (see [SV]) a distinction of flypes according to the orientation near the crossing  $p$  at which the flype is performed. See figure 5. An important observation is that each crossing admits at most one of the types A and B of flypes, and this remains so after applying any sequence on flypes on the diagram.

A *bridge* is a piece of a strand of a knot diagram that passes only crossings from above. The *length* of the bridge is the number of crossings passed by it (*excluding* the initiating and terminating underpass). A *tunnel* is the mirror image of a bridge.

A *wave move* is a replacement of a bridge  $a$  of length  $l_1$  by another one  $b$  of length  $l_2$ . We will assume throughout that  $l_2 \leq l_1$ . The move is (crossing number) *reducing*, if  $l_2 < l_1$  (see figure 6 or [StK] for example), and (crossing number) *preserving*, if  $l_2 = l_1$ . Elsewhere a wave move is also called a  $(l_1, l_2)$ -*pass*. Note that a  $(1, 0)$ -pass is the removal of a nugatory crossing (much like in figure 1, except that  $P$  is flipped), and a  $(1, 1)$ -pass is exactly the factor slide of figure 2 (b). We will usually consider only reducing wave moves, except for the case  $l_1 = l_2 = 1$ . Obviously, such a move works for tunnels instead of bridges and we *will not distinguish between both*, unless clearly stated.

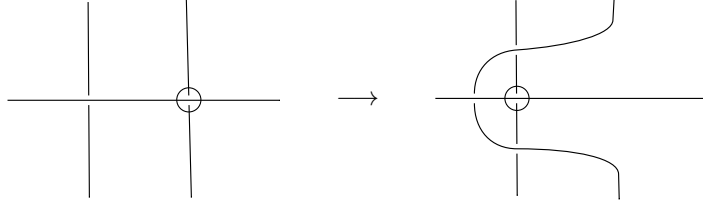


**Figure 6:** Wave-moves. The number of strands on left and right of the shaded circle may vary. It is only important that the parities are equal resp. different, and that the left-outgoing strands are fewer than the right-outgoing ones.

Adams introduced a *tongue move* allowing to build more complicated almost alternating diagrams from a given



one. This move is shown in figure 7. Herein the crossing needed to be switched to obtain an alternating diagram is encircled; we call this crossing *dealternator*.



**Figure 7:** A tongue move between two almost alternating diagrams. The encircled crossing is to be switched to obtain an alternating diagram.

Adams formulated a conjecture, stating how to recognize the unknot in almost alternating diagrams, which was proved recently by Tsukamoto.

**Theorem 2.6** (Tsukamoto [Ts]) An almost alternating unknot diagram is trivializable by tongue moves, flypes, and crossing number reducing Reidemeister I and II moves.

A tongue move is a special type of wave move, preserving the property almost alternating. In that sense theorems 1.1 and 5.1 can be thought of as extending Tsukamoto's result in special diagrams.

For  $\leq 10$  crossings we use the numbering of prime knots of [Ro]. For knots from 11 to 16 crossings our numbering is that of KnotScape [HT]. Latter is reorganized so that non-alternating knots are appended after alternating ones (of the same crossing number), instead of using 'a' and 'n' superscripts.

We write  $!D$  for the *mirror image* of  $D$ , and  $!K$  denotes the mirror image of  $K$ . Clearly  $g(!D) = g(D)$  (and therefore  $\tilde{g}(K) = \tilde{g}(!K)$ ), and  $g(!K) = g(K)$ .

## 2.4 Link polynomials

Let  $X \in \mathbb{Z}[t, t^{-1}]$ . The *minimal* or *maximal degree*  $\min \deg X$  or  $\max \deg X$  is the minimal resp. maximal exponent of  $t$  with non-zero coefficient in  $X$ . Let  $\text{span}_t X = \max \deg_t X - \min \deg_t X$ , the *span* of *breadth* of  $X$ . The coefficient in degree  $d$  of  $t$  in  $X$  is denoted  $[X]_{t^d}$  or  $[X]_d$ . The *leading coefficient*  $\max \text{cf} X$  of  $X$  is its coefficient in degree  $\max \deg X$ . If  $X \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ , then  $\max \deg_{x_1} X$  denotes the maximal degree in  $x_1$ . Minimal degree and coefficients are defined similarly, and of course  $[X]_{x_1^k}$  is regarded as a polynomial in  $x_2^{\pm 1}$ .

The *skein polynomial*  $P$  [F&, LM] is a Laurent polynomial in two variables  $l$  and  $m$  of oriented knots and links and can be defined by being 1 on the unknot and the (skein) relation

$$l^{-1} P \left( \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right) + l P \left( \begin{array}{c} \nwarrow \nearrow \\ \nearrow \searrow \end{array} \right) = -m P \left( \begin{array}{c} \nearrow \nearrow \\ \searrow \searrow \end{array} \right). \quad (3)$$

(The convention differs from [LM] by the interchange of  $l$  and  $l^{-1}$ .) We will denote in each triple as in (3) the diagrams (from left to right) by  $D_+$ ,  $D_-$  and  $D_0$ . For a diagram  $D$  of a link  $L$ , we will use all of the notations  $P(D) = P_D = P_D(l, m) = P(L)$  etc. for its skein polynomial, with the self-suggestive meaning of indices and arguments.

The *Jones polynomial* [J]  $V$ , and (one variable) *Alexander polynomial* [Al]  $\Delta$  are obtained from  $P$  by the substitutions

$$V(t) = P(-it, i(t^{-1/2} - t^{1/2})), \quad (4)$$

$$\Delta(t) = P(i, i(t^{1/2} - t^{-1/2})), \quad (5)$$

hence these polynomials also satisfy corresponding skein relations. (In algebraic topology, the Alexander polynomial is usually defined only up to units in  $\mathbb{Z}[t, t^{-1}]$ ; the present normalization is so that  $\Delta(t) = \Delta(1/t)$  and  $\Delta(1) = 1$ .)

We will use sometimes instead of  $\Delta$  also the *Conway polynomial* [Co]  $\nabla(z)$  with  $\Delta(t) = \nabla(t^{1/2} - t^{-1/2})$ .  $\nabla$  satisfies the skein relation  $\nabla(D_+) - \nabla(D_-) = z\nabla(D_0)$ . Note that  $\max \deg \nabla = 2 \max \deg \Delta$ , which we use in particular to implicitly restate some of the results of [Cr] in the sequel.

The *Kauffman polynomial* [Kf]  $F$  is usually defined via a regular isotopy invariant  $\Lambda(a, z)$  of unoriented links. We use here a slightly different convention for the variables in  $F$ , differing from [Kf, Th] by the interchange of  $a$  and  $a^{-1}$ . Thus in particular we have for a link diagram  $D$  the relation  $F(D)(a, z) = a^{w(D)} \Lambda(D)(a, z)$ , where  $\Lambda(D)$  is the writhe-unnormalized version of the polynomial, given in our convention by the properties

$$\begin{aligned} \Lambda(\times) + \Lambda(\times) &= z (\Lambda(\smile) + \Lambda(\frown)), \\ \Lambda(\bigcirc) &= a^{-1} \Lambda(\mid); \quad \Lambda(\bigcirc) = a \Lambda(\mid), \\ \Lambda(\bigcirc) &= 1. \end{aligned}$$

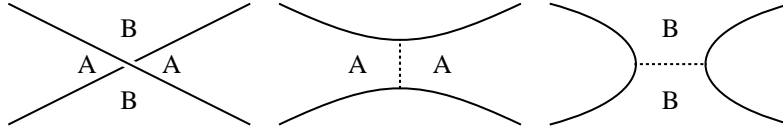
The Jones polynomial  $V$  is obtained from  $F$  (in our convention) by the substitution (see [Kf2, §III])

$$V(t) = F(-t^{3/4}, t^{1/4} + t^{-1/4}).$$

An alternative description of  $V$  is given by the Kauffman bracket in [Kf2], which we recall next. The Kauffman bracket  $[D]$  of a diagram  $D$  is a Laurent polynomial in a variable  $A$ , obtained by summing over all states the terms

$$A^{\#A - \#B} (-A^2 - A^{-2})^{|S| - 1}. \quad (6)$$

A *state* is a choice of splittings of type  $A$  or  $B$  for any single crossing (see figure 8),  $\#A$  and  $\#B$  denote the number of type  $A$  (resp. type  $B$ ) splittings and  $|S|$  the number of (disjoint) circles obtained after all splittings in a state.



**Figure 8:** The A- and B-corners of a crossing, and its both splittings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counterclockwise (resp. clockwise) towards the undercrossing strand. A type A (resp. B) splitting is obtained by connecting the A (resp. B) corners of the crossing. It is useful to put a “trace” of each splitted crossing as an arc connecting the loops at the splitted spot.

The Jones polynomial of a link  $L$  is related to the Kauffman bracket of some diagram  $D$  of  $L$  by

$$V_L(t) = \left( -t^{-3/4} \right)^{-w(D)} [D] \Big|_{A=t^{-1/4}}. \quad (7)$$

Let the *A-state* of  $D$  be the state where all crossings are A-spliced; similarly define the *B-state*. We call a diagram *A-(semi)adequate* if in the A-state no crossing trace (one of the dotted lines in figure 8) connects a loop with itself. We call such a trace a *self-trace*. Similarly we define *B-(semi)adequate*. A diagram is *semiadequate* if it is A- or B-semiadequate, and *adequate* if it is simultaneously A- and B-semiadequate. A link is adequate/semiadequate if it has an adequate/semiadequate diagram.

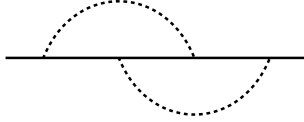
**Theorem 2.7** ([St11]) For a semiadequate knot (or link) diagram  $D$ , we have  $V(D) \neq 1$  (or  $V(D) \neq (-t^{1/2} - t^{-1/2})^{n(D)-1}$ ).

After the conversion (7), we see that the minimal degree of  $t$  to which the sum (6) contributes occurs in the A-state  $A(D)$ . For our subsequent arguments, this degree can be more conveniently written for a knot diagram  $D$  as

$$m(D) := g(D) - \frac{3}{2}c_-(D) + \frac{s(D) - |A(D)|}{2}. \quad (8)$$

If a diagram is  $A$ -adequate, only the  $A$ -state contributes in degree  $m(D)$ , and the coefficient is  $\pm 1$  (see [LT]). In Bae-Morton [BMo], the contribution of the sum of (6) in degree  $m(D)$  for general diagrams  $D$  was studied. The following easy property will be useful.

Call a self-trace *isolated*, if it does not pair up with another self-trace like



**Lemma 2.1** (Bae-Morton [BMo]) If the  $A$ -state of  $D$  has an isolated self-trace, then the contribution of the sum of (6) in degree  $m(D)$  vanishes, that is,  $\min \deg V(D) > m(D)$ .

Let  $F(K) \in \mathbb{Z}[a^{\pm 1}, z]$  be the Kauffman polynomial of a knot  $K$  and  $Q_K(z) = F_K(1, z)$  be the BLMH polynomial [BLM, Ho]. Let further

$$a_-(K) = \max \{m - l : [F(K)]_{a^l z^m} \neq 0\} \quad \text{and} \quad a_+(K) = \max \{m + l : [F(K)]_{a^l z^m} \neq 0\}. \quad (9)$$

It easily follows that  $\max \deg_z F(K) \leq a_+(K) + a_-(K)$ . Moreover, if the r.h.s. is positive, strict inequality holds, because the substitution  $F(i, z) = 1$  shows that  $\max \text{cf}_z F(K)$  cannot be a single monomial (in  $a$ ).

In [Th], Thistlethwaite proves that

$$c_{\pm}(K) \geq a_{\pm}(K), \quad (10)$$

with equality if and only if  $K$  is  $A$  resp.  $B$ -semiadequate. Let us say in the following that a (non-strict) inequality of the form ' $A \geq B$ ' is *sharp* or *exact* if  $A = B$ , and *strict* otherwise (i.e. if  $A > B$ ).

Adequate (in particular alternating) diagrams make both inequalities in (10) simultaneously sharp. It follows from these inequalities that for every non-trivial knot  $K$ ,

$$\max \deg Q(K) \leq \max \deg_z F(K) < c_+(K) + c_-(K). \quad (11)$$

(If  $c_-(K) = c_+(K) = 0$ , then  $K$  is simultaneously positive and negative, and so is trivial; see for example [St6], or the below remarks on the signature.)

Note that for  $P$  and  $F$  there are several other variable conventions, differing from each other by possible inversion and/or multiplication of some variable by some fourth root of unity.

## 2.5 The signature

The *signature*  $\sigma$  is a  $\mathbb{Z}$ -valued invariant of knots and links. Originally it was defined terms of Seifert matrices [Ro]. We have that  $\sigma(L)$  has the opposite parity to the number of components of a link  $L$ , whenever the *determinant*  $\det(L) = |\Delta_L(-1)| \neq 0$ . This in particular always happens for  $L$  being a knot (since  $\Delta_L(-1)$  is always odd in this case), so that  $\sigma$  takes only even values on knots. Most of the early work on the signature was done by Murasugi [Mu5], who showed several properties of this invariant.

Then, for links  $L_{\pm, 0}$  with diagrams as in (3), we have

$$\sigma(L_+) - \sigma(L_-) \in \{0, 1, 2\} \quad (12)$$

$$\sigma(L_{\pm}) - \sigma(L_0) \in \{-1, 0, 1\}. \quad (13)$$

(Note: In the first property one can also have  $\{0, -1, -2\}$  instead of  $\{0, 1, 2\}$ , since other authors, like Murasugi, take  $\sigma$  to be with opposite sign. Thus (12) not only defines a property, but also specifies our sign convention for  $\sigma$ .)

Further, Murasugi found the following important relation between  $\sigma(K)$  and  $\det(K)$  for a knot  $K$ .

$$\begin{aligned} \sigma(K) \equiv 0(4) &\iff \det(K) \equiv 1(4) \\ \sigma(K) \equiv 2(4) &\iff \det(K) \equiv 3(4) \end{aligned} \quad (14)$$

If  $\det = 1$ , then even  $8 \mid \sigma$ , because of the property of signatures of unimodular quadratic forms.

These conditions, together with the initial value  $\sigma(\bigcirc) = 0$  for the unknot, and the additivity of  $\sigma$  under split union (denoted by  $\sqcup$ ) and connected sum (denoted by  $\#$ )

$$\sigma(L_1 \# L_2) = \sigma(L_1 \sqcup L_2) = \sigma(L_1) + \sigma(L_2),$$

allow one to calculate  $\sigma$  for most links (incl. all knots). The following further property is very useful:  $\sigma(!L) = -\sigma(L)$ , where  $!L$  is the mirror image of  $L$ .

If  $K$  is a positive knot, then  $\sigma(K) > 0$  [CG]. Przytycki's concern was to improve and extend this result. In particular,  $\sigma(K) \geq 4$  if  $g(K) \geq 2$ ; also  $\sigma(K) > 0$  if  $K$  is almost positive or 2-almost positive, except a twist knot. See [St3, St7, St2] or §6 below for written (though not identical) proofs.

## 2.6 Braid index and skein polynomial

The *braid group*  $B_n$  on  $n$  strands (or strings) is considered to be generated by the Artin *standard generators*  $\sigma_i$  for  $i = 1, \dots, n-1$ . These are subject to relations of the type  $[\sigma_i, \sigma_j] = 1$  for  $|i-j| > 1$ , which we call *commutativity relations* (the bracket denotes the commutator) and  $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$ , which we call *Yang-Baxter* (or shortly YB) relations.

The *braid index*  $b(L)$  of a link  $L$  is the smallest number of strands of a braid  $\beta$  whose closure  $\hat{\beta}$  is  $L$ . See [Mo, FW, Mu4]. (Alexander's theorem asserts that  $L = \hat{\beta}$  for some braid  $\beta$ .)

In [Mo, FW] it was proved that

$$\text{mwf}(L) := \frac{1}{2} \text{span}_L P(L) + 1 \leq b(L), \quad (15)$$

the *Morton-Williams-Franks* (MWF) *inequality*. Since we will need the left-hand side of this inequality later, let us write for it  $\text{mwf}(L)$  and call it the *Morton-Williams-Franks bound* for  $b(L)$ . The inequality (15) results from two other inequalities, due to Morton, namely that for a diagram  $D$ , we have

$$1 - s(D) + w(D) \leq \min \deg_l P(D) \leq \max \deg_l P(D) \leq s(D) - 1 + w(D). \quad (16)$$

Williams-Franks showed these inequalities for the case of braid representations. Later it was observed from the algorithm of Yamada [Ya] and Vogel [Vo] that the braid version is actually equivalent to, and not just a special case of, the diagram version. (These algorithms allow to turn any diagram  $D$  into a braid diagram without altering  $s(D)$  and  $w(D)$ .) Nonetheless we will refer below to (16) as 'Morton's inequalities'.

These inequalities were later improved in [MP] in a way that allows to settle the braid index problem for many links (see §7 or also [Oh]). For this purpose, Murasugi-Przytycki developed the concept of *index of a graph*. We recall some main points of Murasugi-Przytycki's work, referring to [MP] for further details, and cautioning to the correction explained in §7.1–7.3 below.

**Definition 2.2** Let  $G$  be a signed graph (each edge carries a sign  $+$  or  $-$ ). For a vertex  $v$  in  $G$  let  $G_v = G/v$  be the graph obtained by contracting the *star*  $\text{star } v$  of  $v$ , that is, the set of edges of which  $v$  is one of the endpoints. Let  $G^v$  be the graph obtained from  $G$  by deleting all these edges and  $v$ .

We say that  $v$  is a *cut vertex* if  $G^v$  is disconnected. A cut vertex decomposes  $G$  into a *block sum* or *join*  $G_1 * G_2$  of two graphs  $G_{1,2}$ . If  $G = G_1 * \dots * G_n$  and all  $G_i$  have no cut vertex, the  $G_i$  will be called *block components* or *join factors* of  $G$ .

**Definition 2.3** We define (recursively) a sequence of edges  $\mu = (e_1, \dots, e_n)$  to be *independent* in a graph  $G$ , if the following conditions are satisfied.

1. The empty (edge) sequence is independent per definition.
2. Let  $e_1$  connect vertices  $v_{1,2}$ . Then we demand that  $e_1$  is *simple*, i.e. there is no other edge connecting  $v_{1,2}$ , and that  $e_2, \dots, e_n$  is independent in (one of)  $G_{v_1}$  or  $G_{v_2}$  (i.p.  $e_2, \dots, e_n$  are disjoint from the set of edges incident to  $v_1$  or  $v_2$  resp.).

An *independent set* is a set of edges admitting an ordering as an independent sequence.

The *index*  $\text{ind}(G)$ , resp. *positive index*  $\text{ind}_+(G)$  and *negative index*  $\text{ind}_-(G)$  of  $G$  are defined as the maximal length of an independent edge set (or sequence), resp. independent positive or negative edge set/sequence in  $G$ . A sequence is *maximal independent* if it realizes the index of  $G$ .

(This index is to be kept strictly apart from the index of a Seifert circle of §2.2. Both are not unrelated, but we do not investigate about their relation, and use them in quite separate contexts.)

Now to each link diagram  $D$  we associate its *Seifert graph*  $G = \Gamma(D)$ , which is a plane bipartite signed graph. It consists of a vertex for each Seifert circle in  $D$  and an edge for each crossing, connecting two Seifert circles. Each edge is signed by the skein sign of the crossing it represents. We will for convenience sometimes identify crossings/Seifert circles of  $D$  with edges/vertices of  $G$ . Let also  $\text{ind}_{(\pm)}(D) = \text{ind}_{(\pm)}(\Gamma(D))$ .

**Proposition 2.1** (see [MP, (8.4) and (8.8)]) If  $D$  is a diagram of an oriented link  $L$ , then

$$\max \deg_l P(L) \leq w(D) + s(D) - 1 - 2\text{ind}_+(D) \quad (17)$$

$$\min \deg_l P(L) \geq w(D) - s(D) + 1 + 2\text{ind}_-(D) \quad (18)$$

$$b(L) \leq \text{mpb}(D) := s(D) - \text{ind}(D). \quad (19)$$

For any diagram  $D$ , we have

$$\text{ind}_+(D) + \text{ind}_-(D) \geq \text{ind}(D). \quad (20)$$

For alternating (and more generally homogeneous [Cr]) diagrams  $D$  equality holds, because each join factor of  $\Gamma(D)$  contains only edges of the same sign. This implies that if in such diagrams (17), (18) are sharp, then (15) and (19) become sharp, too.

It is clear that one can reconstruct  $D$  from  $\Gamma(D)$  (when latter is given in a planar embedding). If  $\Gamma(D)$  has block components  $G_i$ , then we call the diagrams  $D_i$  with  $\Gamma(D_i) = G_i$  the *Murasugi atoms* of  $D$ . An alternative way to specify  $D_i$  is to say that they are the prime factors of the Murasugi summands of  $D$  (see definition 3.1 below, and [QW]).

The following theorem of Murasugi-Przytycki is important:

**Theorem 2.8** (theorem 2.5 in [MP]) The index is additive under block sum of bipartite graphs, i.e.  $\text{ind}(G_1 * G_2) = \text{ind}(G_1) + \text{ind}(G_2)$  if  $G_{1,2}$  are bipartite.

We will treat below, simultaneously to (the sharpness of) MWF, the following conjecture of theirs.

**Conjecture 2.1** (Murasugi-Przytycki) If  $D$  is an alternating diagram of a link  $L$ , then  $b(L) = \text{mpb}(D)$ .

## 2.7 Genus generators

Now let us recall, from [St4, St2], some basic facts concerning knot generators of given genus. An explanation is given also in section 5.3 in [Cr2]. (There are several equivalent forms of these definitions, and we choose here one that closely leans on the terminology of Gauß diagrams; see for example [St].) We will also set up some notations and conventions used below. The situation for links is discussed only briefly here, and in much more detail in §3.

**Definition 2.4** We call two crossings  $p, q$  in a knot diagram *linked*, and write  $p \cap q$ , if passing their crossingpoints along the orientation of  $D$  we have the cyclic order  $pqqp$ , and not  $ppqq$ .

**Definition 2.5** Let  $D$  be a knot diagram, and  $p$  and  $q$  be crossings.

- (i) We call  $p$  and  $q$  (*twist*) *equivalent*,  $q \simeq p$ , if for all  $r \neq p, q$  we have  $r \cap p \iff r \cap q$ .

- (ii) We call  $p$  and  $q$   $\sim$ -equivalent and write  $p \sim q$  if  $p$  and  $q$  are equivalent and  $p \not\cap q$ .
- (iii) Similarly  $p$  and  $q$  are called  $\approx$ -equivalent,  $p \approx q$ , if  $p$  and  $q$  are equivalent and  $p \cap q$ .
- (iv) Finally, call two crossings  $p$  and  $q$  *Seifert equivalent*, if they connect the same two Seifert circles.

Let us record the following easy but useful observation.

**Lemma 2.2**  $\approx$ -equivalent crossings are Seifert equivalent. The converse is true in special diagrams.  $\square$

**Definition 2.6** A  $\sim$ -equivalence class consisting of one crossing is called *trivial*, a class of more than one crossing *non-trivial*. A  $\sim$ -equivalence class is *reduced* if it has at most two crossings; otherwise it is *non-reduced*.

**Definition 2.7** Let  $t(D)$  be the number of  $\sim$ -equivalence classes of  $D$ . For  $i = 1, \dots, t(D)$  let  $t_i(D)$  be the number of crossings in the  $i$ -th  $\sim$ -equivalence class. Then for  $i = 1, \dots, t(D)$  and  $j = 1, \dots, t_i(D)$  let  $p(D, i, j)$  be the  $j$ -th crossing in the  $i$ -th  $\sim$ -equivalence class of  $D$ .

**Definition 2.8** A  $\vec{t}_2$  move or twist at a crossing  $x$  in a diagram  $D$  is a move, which creates a pair of  $\sim$ -equivalent crossings to  $x$ . (This is well-defined up to flypes.) We will call it positive or negative, depending on whether it acts on a crossing of the according sign.

**Definition 2.9** An alternating diagram  $D$  is called  $\vec{t}_2$  *irreducible* or *generating diagram*, if all  $\sim$ -equivalence classes are reduced, that is,  $t_i(D) \leq 2$  for  $i = 1, \dots, t(D)$ . An alternating knot  $K$  is called *generator* if some of its alternating diagrams is generating. The diagrams obtained from  $D$  by  $\vec{t}_2$  moves and crossing changes form the *sequence* or *series*  $\langle D \rangle$  of  $D$ .

(In many cases it will be useful to restrict oneself in  $\langle D \rangle$  to alternating or positive diagrams.)

Observe that, since  $\sim$ -equivalence is invariant under flypes, theorem 2.5 implies that some alternating diagram of  $K$  is generating if and only if all its alternating diagrams are so.

A flype in figure 4 is called *trivial* if one of the tangles  $P, Q$  contains only crossings equivalent to the crossing admitting the flype. A flype is of type A if the strand orientation is so that strands on the left/right side of each tangle are directed equally w.r.t. the tangle (i.e. both enter or both exit). Otherwise it is a flype of type B. Compare [SV]. So the property a crossing to admit a type A resp. type B flype is invariant of the  $\approx$  resp.  $\sim$ -equivalence class.

**Theorem 2.9** ([St4]) There exist only finitely many generators of given genus. All diagrams of that genus can be obtained from diagrams of these generators, under  $\vec{t}_2$  twists, flypes, and crossing changes.

The finiteness of generators, together with the Flyping theorem [MT], shows

**Theorem 2.10** (see [St4]) Let  $a_{n,g}$  be the number of prime alternating knots  $K$  of genus  $g(K) = g$  and crossing number  $c(K) = n$ . Then for  $g \geq 1$

$$\sum_n a_{n,g} x^n = \frac{R_g(x)}{(x^{p_g} - 1)^{d_g}},$$

for some polynomial  $R_g \in \mathbb{Z}[x]$ , and  $p_g, d_g \in \mathbb{N}$ . Alternatively, this statement can be written also in the following form: there are numbers  $p_g$  (period),  $n_g$  (initial number of exceptions) and polynomials  $P_{g,1}, \dots, P_{g,p_g} \in \mathbb{Q}[n]$  with  $a_{n,g} = P_{g,n \bmod p_g}(n)$  for  $n \geq n_g$ .

This was explained roughly in [St4], and then in more detail in [SV], where we made effort to characterize the leading coefficient of these polynomials  $P_{g,i}$ . Even if the polynomials vary with a very large period  $p_g$ , the leading coefficients depend only on the parity of  $n$ . The degrees of all  $P_{g,i}$  are also the same, and equal to 1 less than the maximal number of  $\sim$ -equivalence classes of diagrams of canonical genus  $g$ , with the exception  $g = 1$ , in which case this degree is 1 or 2 depending on whether  $i$  is even or odd.

In practice (in particular as we will see below) it is important to obtain the list of generators for small genus. Genus one is easy, and also observed independently.

**Theorem 2.11** ([St4]; see also [Ru]) There are two generators of genus one, the trefoil and figure-8-knot.

Genus two and three require much more work. For suggestive reasons, it is sufficient to find prime diagrams, and by theorem 2.1, prime generators.

**Theorem 2.12** ([St2]) There are 24 prime generators of genus two,  $5_1, 6_2, 6_3, 7_5, 7_6, 7_7, 8_{12}, 8_{14}, 8_{15}, 9_{23}, 9_{25}, 9_{38}, 9_{39}, 9_{41}, 10_{58}, 10_{97}, 10_{101}, 10_{120}, 11_{123}, 11_{148}, 11_{329}, 12_{1097}, 12_{1202}$ , and  $13_{4233}$ , and 4017 prime generators of genus 3.

A classification (by means of obtaining the list of prime generators) for genus  $g = 4$  is also possible, and will be explained below, as well as a general statement in theorem 3.1.

It follows from theorem 2.5 that the series of different (alternating) diagrams of the same generating knot are equivalent *up to mutations*. When using tests involving the polynomial invariants, which are invariant under mutations, it is legitimate that *a priori* we fix a *single specific diagram  $D$  for each generator  $K$* , and work only with (the series of) this diagram. Then we write  $t(K) = t(D)$ ,  $t_i(K) = t_i(D)$  and  $p(K, i, j) = p(D, i, j)$ , and speak of the sequence or series of  $K$ . With some simple observations we will be able to comfort us with the assumption in §7, while in §5 more care is needed.

Let us parametrize diagrams in the series of a generator  $K$  as  $K(x_1, \dots, x_l)$ , where  $l = t(K)$ , the  $l$   $\sim$ -equivalence classes of  $K$  are ordered in some fixed way, and  $x_i$  are defined as follows.

For a trivial  $\sim$ -equivalence (i.e.  $t_i = 1$ ),  $x_i \geq 1$  means a  $\sim$ -equivalence class of  $2x_i - 1$  positive crossings, or alternatively, the result of applying  $(x_i - 1)$   $\vec{t}_2$ -moves to a single positive crossing in the generator. If  $x_i \leq 0$ , then we have a  $\sim$ -equivalence class of  $1 - 2x_i$  negative crossings, that is, a negative crossing with  $(-x_i)$   $\vec{t}_2$ -moves applied.


For a  $\sim$ -equivalence class of  $t_i = 2$  crossings,  $x_i > 0$  means a  $\sim$ -equivalence class of  $2x_i$  positive crossings, or alternatively, the result of applying  $(x_i - 1)$   $\vec{t}_2$ -moves to one of the two positive crossings in the generator.  $x_i = 0$  means that the two crossings have opposite sign, that is, form a trivial clasp after flypes.  $x_i < 0$  means  $-2x_i$  negative crossings in the  $\sim$ -equivalence class, that is, the result of  $(-1 - x_i)$   $\vec{t}_2$ -moves on one of the negative two crossings in the generator.


This convention will remain valid for the rest of the paper. Note that it implies that we discard diagrams with crossings of different sign within the same  $\sim$ -equivalence class (unless  $x_i = 0$  and  $t_i = 2$ ). Such diagrams have a trivial clasp after flypes, and are of little interest. We will call a  $\sim$ -equivalence class *positive* or *negative* depending on the sign of its crossings.

Turning to *link* diagrams, the  $\vec{t}_2$  twist (or  $\vec{t}_2$  move) is given up to mirroring by

$$\begin{array}{c} \nearrow \quad \nwarrow \\ \searrow \quad \nearrow \end{array} \rightarrow \begin{array}{c} \nearrow \quad \nearrow \\ \searrow \quad \searrow \end{array} \quad (21)$$

We call diagrams that cannot be reduced by flypes and inverses of the move (21) *generating* or  $\vec{t}_2$ -*irreducible*. We know (in the case of knots, but we will extend this result below to links) that reduced knot diagrams of given genus  $g$  (with  $1 - \chi = 2g$ ) decompose into finitely many equivalence classes under  $\vec{t}_2$  twists and their inverses. We call these collections of diagrams *series*.

Recall that in [St4] we called two crossings  $\sim$ -*equivalent*, if after a sequence of flypes they can be made to form a *reverse clasp* ; it is an exercise to check that this is an equivalence relation. Similarly we call (see [St2])

two crossings  $\approx$ -*equivalent*, if after a sequence of flypes they can be made to form a *parallel clasp* . We observed (see [St4, St2, STV]) that  $\sim$ - and  $\approx$ -equivalent crossings of a knot diagram are linked with the same set of other crossings in the diagram, so we have an equivalence to the previous definition in the case of knots.

## 2.8 Knots vs. links

To conclude our setup, let us make more precise the separation between knots and links in the following work.

The methods developed in this paper put into prospect to adapt results about ‘ $k$ -almost positive knots’ and ‘alternating knots of genus  $k$ ’ to statements that apply to ‘ $\tilde{k}$ -almost positive  $l$ -component links’ and ‘alternating  $l$ -component links of genus  $\tilde{k}$ ’ for  $\tilde{k} + l \leq k + 1$ . We have largely waived on treating links for a technical reason: the way we designed our computation (see similarly remark 5.1).

The results where links are covered are those in §3 (here the inclusion of links is essential and will be used elsewhere) and §7.5. They rely on purely theoretical arguments, and computation is not needed. To some extent, we managed to adapt computations to the link case, as is shown in theorem 9.6. However, for other parts more effort will be needed. We also expect that some statements for links would look less pleasant than for knots.

## 3 The maximal number of generator crossings and $\sim$ -equivalence classes

### 3.1 Generator crossing number inequalities

As explained in the introduction, we must first spend some effort in estimates for the crossing number of generators of given canonical Euler characteristic. These estimates turn out to be rather sharp, and are a consequence of a detailed study of the special diagram algorithm of Hirasawa [Hi2] and myself [St9, §7].

**Theorem 3.1** In a connected link diagram  $D$  of canonical Euler characteristic  $\chi(D) \leq 0$  there are at most

$$t(D) \leq \begin{cases} -3\chi(D) & \text{if } \chi(D) < 0 \\ 1 & \text{if } \chi(D) = 0 \end{cases} \quad (22)$$

$\sim$ -equivalence classes of crossings. If  $D$  is  $\tilde{l}_2'$ -irreducible and has  $n(D)$  link components, then

$$c(D) \leq \begin{cases} 4 & \text{if } \chi(D) = -1 \text{ and } n(D) = 1, \\ 2 & \text{if } \chi(D) = 0, \\ -6\chi(D) & \text{if } \chi(D) < 0 \text{ and } n(D) = 2 - \chi(D), \\ -5\chi(D) + n(D) - 3 & \text{else.} \end{cases} \quad (23)$$

Thus we settle the problem to determine the maximal crossing number of a generator for knots.

**Corollary 3.1** The maximal crossing number of a knot generator of genus  $g \geq 2$  is  $10g - 7$ .

**Proof.** We know from [SV] that for any  $g \geq 2$ , there are examples of generator diagrams with  $10g - 7$  crossings.  $\square$

The inequality (22) is also optimal, and we call the generating diagrams  $D$  that make (22) exact *maximal generating diagrams* and their knots/links *maximal generators*. For knots these generators were studied in detail in [SV].

The main idea behind the description of diagrams of given canonical Euler characteristic was to show that they decompose into finitely many equivalence classes under  $\tilde{l}_2'$  twists and their inverses. In [St4] we showed this only for knots, but the case of links can be easily recurred to it. For a diagram  $D$  of a link  $L$  with  $n(L)$  components, one applies  $n - 1 = n(L) - 1$  moves, which replace a positive/negative crossing of two different components by a parallel positive/negative clasp:

$$\begin{array}{ccc} \begin{array}{c} \nearrow \\ \searrow \end{array} & \longrightarrow & \begin{array}{c} \nearrow \quad \searrow \\ \searrow \quad \nearrow \end{array}, & \begin{array}{c} \nearrow \\ \searrow \end{array} & \longrightarrow & \begin{array}{c} \nearrow \quad \searrow \\ \searrow \quad \nearrow \end{array}. \end{array} \quad (24)$$

We call this procedure below a *clasping*. Thus  $D$  can be transformed into a knot diagram  $D'$ , with  $1 - \chi(D') = n - \chi(D)$ .



Although this simple argument establishes the picture qualitatively, it is not useful for an optimal estimate, and neither was our original approach in [St4] for knots. Then, in subsequent work [STV, SV] we established a relation between canonical Seifert surfaces for knot diagrams and 1-vertex triangulations of surfaces, and obtained some partial information (certainly much better than in [St4]) on the maximal number of crossings and  $\sim$ -equivalence classes of  $\bar{\ell}_2$ -irreducible knot diagrams of given canonical genus. Unfortunately, this method does not extend pleasantly to links. We will thus introduce now a different approach, which is entirely knot theoretic and circumvents this problem. (Contrarily, there are insights of the old approach, which will also be used later, but which cannot be recovered.) This approach originates from an algorithm, first found by Hirasawa [Hi], to make any link diagram into a special one without altering the canonical Euler characteristic (in fact, even its isotopy type).

We will apply this algorithm to prove the above result theorem 3.1, which is likely in its optimal and most general form. Before the proof of theorem 3.1, first we discuss Hirasawa's algorithm.

### 3.2 An algorithm for special diagrams

**Definition 3.1** Seifert circles in an arbitrary diagram are called *non-separating*, if they have empty interior or exterior; the others are called *separating*. A diagram is called *special* if all its Seifert circles are non-separating. Any link diagram decomposes along its separating Seifert circles as the *Murasugi sum* ( $*$ -product) of special diagrams (see [Cr, §1]).

In [BZ] it was proved that each link has a special diagram by a procedure how to turn any given diagram of the link into a special one. However, the procedure in this proof alters drastically the initial diagram and offers no reasonable control on the complexity (canonical genus and crossing number) of the resulting special diagram. A much more economical procedure was given by Hirasawa [Hi] and rediscovered a little later independently in [St9, §7]. Hirasawa's move consists of laying a part of a separating Seifert circle  $s$  along itself in opposite direction (we call this move *rerouting*; it is the opposite of the wave move in figure 6), while changing the side of  $s$  dependently on whether interior or exterior adjacent crossings to  $s$  are passed. See figure 9.

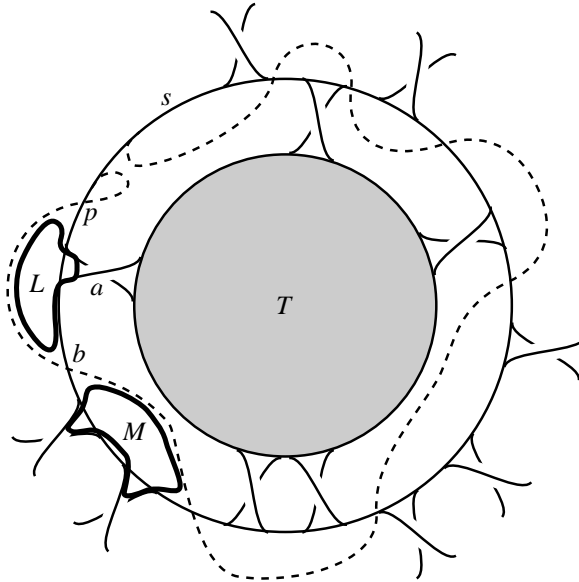


Figure 9

The move of [St9] is similar, only that this type of rerouting is applied to the Seifert circle connected to  $s$  by an crossing  $c$  exterior to  $s$ . This move lowers the canonical Euler characteristic by two, but by properly choosing to

reroute the strand above or below the rest of the diagram (that is, such that it passes all newly created crossings as over- or undercrossings), one obtains a trivial parallel clasp involving  $c$ , whose deletion raises the canonical Euler characteristic back by two. Then we obtain an instance of Hirasawa's move.

Hereby, unlike in Hirasawa's original version of his algorithm, we take the freedom to alter the signs of the new crossings, as far as the isotopy type of the link, but *not* necessarily of the canonical Seifert surface is preserved. It is of importance to us only that the canonical Euler characteristic of the diagram is preserved. We assume that this freedom is given throughout the rest of this section.

Hirasawa's algorithm is very economical – the number of new crossings added is linearly bounded in the crossing number, and even in the canonical Euler characteristic of the diagram started with. (Note, for example, that the braid algorithms of Yamada [Ya] and Vogel [Vo] have quadratic growth.)

Using Hirasawa's algorithm we prove now

**Lemma 3.1** Any connected link diagram  $D$  of maximal number of  $\sim$ -equivalence classes of crossings for given canonical Euler characteristic is special. Moreover, this is also true for any  $\tilde{l}_2$ -irreducible link diagram of maximal crossing number, except if  $\chi(D) = -1$  and  $D$  is the 4-crossing (figure-8-knot) diagram.

The following notions will be of particular importance in the proof:

**Definition 3.2** A *region* of a link diagram is a connected component of the complement of the (plane curve of) the diagram. An *edge* of  $D$  is the part of the plane curve of  $D$  between two crossings (clearly each edge bounds two regions). At each crossing  $p$ , exactly two of the four adjacent regions contain a part of the Seifert circles near  $p$ . We call these the *Seifert circle regions* of  $p$ . The other two regions are called the *non-Seifert circle regions* of  $p$ . If the diagram is special, each Seifert circle coincides with (the boundary of) some region. We call the regions accordingly Seifert circle regions or non-Seifert circle regions (without regard to a particular crossing).

Note that two crossings are  $\sim$ -equivalent iff they share the same pair of non-Seifert circle regions.

**Definition 3.3** We call two crossings  $a$  and  $b$  in a diagram  $D$   *neighbored*, if they belong to a reversely oriented primitive Conway tangle in  $D$ , that is, there are crossings  $c_1, \dots, c_n$  with  $a = c_1$  and  $b = c_n$ , such that  $c_i$  and  $c_{i+1}$  form a reverse clasp in  $D$ .

This is a similar definition to  $\sim$ -equivalence, but with no flypes allowed. Thus the number of  $\sim$ -equivalence classes of a diagram is not more than the number of neighbored equivalence classes of the same diagram, or of any flyped version of it.

**Proof of lemma 3.1.** We consider first the second statement, that is, the one for the crossing number of  $\tilde{l}_2$ -irreducible link diagrams.

To show this, take a non-special reduced diagram  $D$  (not necessarily a generator). Apply flypes so that all  $\sim$ -equivalent crossings are neighbored equivalent. (This is possible, because flyping in a flyping circuit is independent from the other ones. For the definition of flyping circuits and related discussion see [ST, §3]. Alternatively one can consult Lackenby's paper [La], where such diagrams are called 'twist reduced'.)

We will show now that the application of Hirasawa moves and appropriate flypes augments the crossing number of the generator, in whose series the diagram lies, and preserves the condition that  $\sim$ -equivalent crossings are neighbored equivalent. The flypes we will apply always reduce the number of separating Seifert circles, and so does any Hirasawa move. Thus we will be done by induction on the number of separating Seifert circles.

Clearly, one can assume that  $D$  is prime. The composite case follows easily from the prime one.

Consider again the picture of the Hirasawa move, figure 10. Here we avoid the creation of the nugatory crossing  $p$  in figure 9. Let  $s$  be the separating Seifert circle of  $D$  on which the move is performed.

We distinguish three cases according to  $\text{ind}(s)$ .

**Case 1.** If  $\text{ind}(s) = 1$ ,  $D$  is not prime.

**Case 2.** Now assume that  $\text{ind}(s) \geq 3$ . Then  $s$  has  $\text{ind}(s)$  adjacent regions from inside and outside. In figure 10, the inner regions are called  $A, A_1, A_2$  and  $A_3$ , and the outer regions are called  $B, B_1, B_2$  and  $B_3$ . The Hirasawa move splits up from every such region  $R$  a small part  $R'$ , containing a new Seifert circle. (In figure 10 four of the 8 such  $R'$  are displayed.) Then it joins an inner and outer region (here  $A$  and  $B$ ) to a new region we call  $AB$ . It adds  $2\text{ind}(s) - 1$  crossings. We call these crossings new, the others, existing already before the move, old. Call the new diagram  $D'$ .

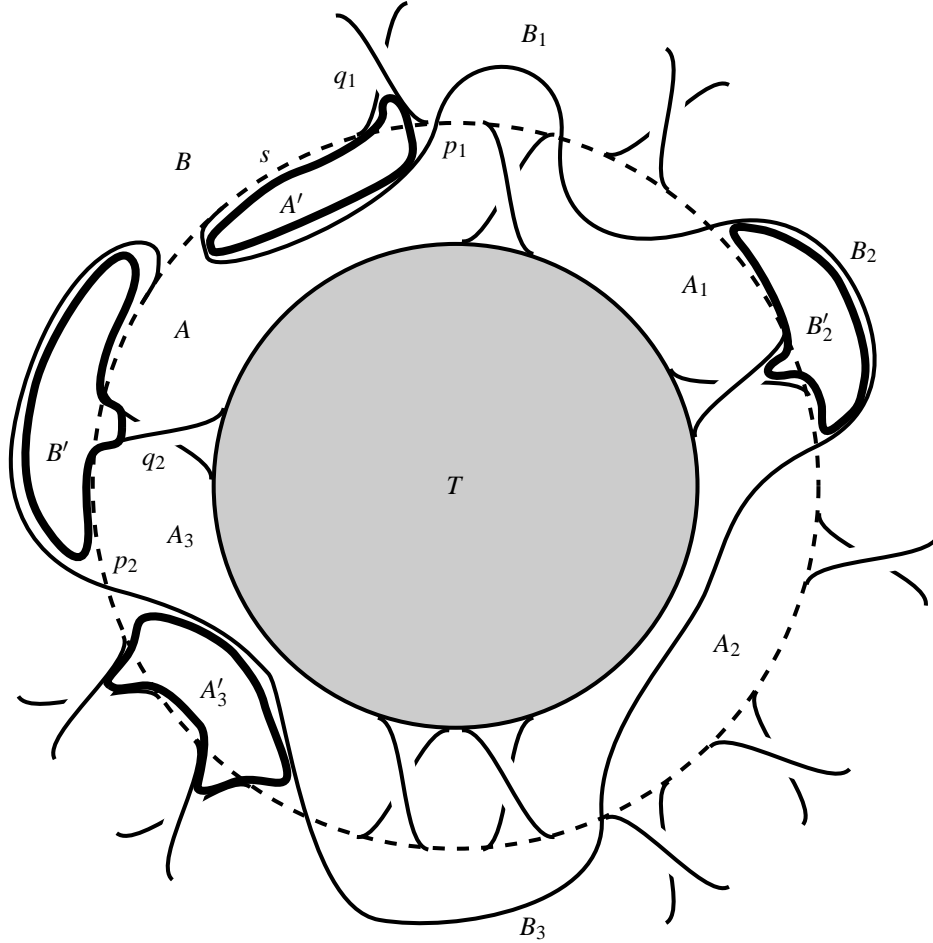


Figure 10

We show that  $D'$  is reduced. Assume that  $p$  is a nugatory crossing in  $D'$ . Then its two non-Seifert circle regions coincide. If  $p$  is an old crossing, the only way to make them distinct by undoing the Hirasawa move is if they are  $A$  and  $B$ . However, the non-Seifert circle regions of any crossing in  $D$  lie either both inside or both outside of  $s$ . If  $p$  is new, then we see directly that the non-Seifert circle regions lie one inside and one outside of  $s$ , and hence are also distinct.

Now we examine how the  $\sim$ -equivalence relation has been altered under the Hirasawa move.

Assume  $p \sim q$  in  $D'$ . Then  $p$  and  $q$  have the same pair of non-Seifert circle regions. (Since no crossing in  $D'$  is nugatory, the two non-Seifert circle regions are always distinct.)

- 1) Assume first  $p$  and  $q$  are old crossings in  $D'$  and  $p \sim q$ . Then the non-Seifert circle regions at  $p$  and  $q$  have remained the same when undoing the Hirasawa move (possibly the small Seifert circle part are removed), except that the region  $AB$  has been separated. Thus  $p \sim q$  in  $D$  also, except if there is a region  $C$  in  $D$  such

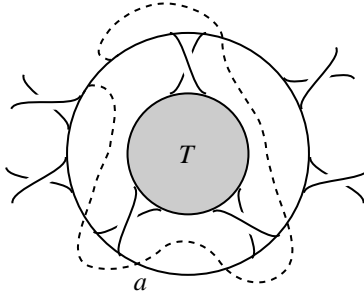
that the non-Seifert circle regions at  $p$  in  $D$  are  $A$  and  $C$ , and the non-Seifert circle regions at  $q$  are  $B$  and  $C$ . However, the non-Seifert circle regions of any crossing lie either both inside or both outside of  $s$ . Since  $A$  and  $B$  lie on different sides of  $s$ ,  $C$  cannot exist. Thus  $p \sim q$  in  $D$ .

- 2) If  $p$  and  $q$  are both new, then the non-Seifert circle regions can be explicitly given among the inner and outer adjacent regions of  $s$  (in our example the  $A$ s and  $B$ s). It is easy to see that from the definition of the Hirasawa moves, all pairs  $(A_i, B_i)$  and  $(A_i, B_{i+1})$  occurring this way are distinct. (Here one needs that  $D$  is prime; some of the  $A_i$  or  $B_i$  alone may coincide!) Thus  $p \not\sim q$ .
- 3) Finally consider the case  $p$  new,  $q$  old. The two non-Seifert circle regions of  $p$  lie one in- and one outside of  $s$  in  $D$ . By the argument with the region  $C$  in case 1), we have  $p \sim q$  only if  $p$  and  $q$  are adjacent to the new region  $AB$  in  $D'$ . From the description of pairs of non-Seifert circle regions of the new crossings and primeness of  $D$ , as mentioned in case 2), there are two such crossings  $p$ . They are the first and last on the newly created strand. They are named  $p_1$  and  $p_2$  in figure 10. They are unique because of the argument in case 2).

We proved that the only possible newly created pairs of  $\sim$ -equivalent crossings are  $(p_1, q_1)$  and  $(p_2, q_2)$ , in which  $q_1$  and  $q_2$  are taken from two fixed (distinct)  $\sim$ -equivalence classes in  $D$ .

The addition of a crossing to a  $\sim$ -equivalence class can reduce the crossing number of the underlying generator by at most 2 (which occurs if the crossing added is of odd number in its  $\sim$ -equivalence class). Since  $2\text{ind}(s) - 1 \geq 5$  new crossings are added, it follows that the crossing number of the generator goes up under a Hirasawa move.

**Case 3.** Consider finally the slightly tricky case  $\text{ind}(s) = 2$ . Now  $2\text{ind}(s) - 5 < 0$  so that we cannot argue as before. Let  $n_i(s)$  and  $n_o(s)$  be the number of inner and outer crossings adjacent to  $s$  resp. Clearly  $n_i(s), n_o(s) \geq \text{ind}(s) = 2$ . If  $n_i(s) = n_o(s) = 2$ , the two internal and external crossings of  $s$  are  $\sim$ -equivalent. Then by assumption they are also neighbored equivalent, and then we have the figure-8-knot diagram of genus one, which we excluded. Thus assume w.l.o.g. (up to  $S^2$ -moves) that  $n_i(s) \geq 3$ . Then we apply a *modified Hirasawa index-2-move*:



(25)

This means, we create an additional trivial clasp in one of the inner groups of at least 2 crossings.

It is easy to see that this move augments the number of generator crossings (at least) by 1.

The condition that  $\sim$ -equivalent crossings are neighbored equivalent is not necessarily preserved under the Hirasawa move. However, using flypes (which preserve the number of crossings and  $\sim$ -equivalence classes), one can again reestablish it. It is an easy observation that the flypes needed reduce the number of separating Seifert circles.

Then iterate Hirasawa moves (and possible flypes), until  $D$  has become special. Then we know, by this argument, that the crossing number of the generator of this special diagram is strictly greater than this of  $D$ . This proves the second assertion in the lemma.

The proof of the first part of the assertion goes along similar lines, but is simpler. Since we showed that the number of  $\sim$ -equivalence classes raises by  $2\text{ind}(s) - 3$  under each Hirasawa move, it is sufficient to assume  $\text{ind}(s) \geq 2$ . This makes unnecessary the flying argument, the need to care about  $\sim$ -equivalent crossings being neighbored equivalent, or about  $\text{ind}(s) = 2$ , and eliminates the exception of the 4-crossing diagram.  $\square$

**Corollary 3.2** For given  $n$  and  $\chi$ , except  $(n, \chi) = (1, -1)$ , a prime generator diagram  $D$  of maximal crossing number is special, and has maximal number of  $\sim$ -equivalence classes.

**Proof.** That  $D$  is special follows from lemma 3.1. If  $D$  is special and has not the maximal number of  $\sim$ -equivalence classes, then by the work of [SV] we know that its unbisected Seifert graph  $G'$  is either not trivalent or not 3-connected. (The argument there was applied for knots, but this condition was not used.) Latter case reduces to former, since if  $G'$  is not 3-connected, one can apply a flype on  $D$  to have a vertex of valence  $\geq 4$  in  $G'$ . In [SV] we argued (under the exclusion of cut vertices), that one can apply a decontraction on such a vertex, such that the created clasp is a new  $\sim$ -equivalence class (and among the previous  $\sim$ -equivalence classes no identifications occur). Then the crossing number of the generator is also augmented.  $\square$

### 3.3 Proof of the inequalities

The lemma does the main part of the proof of theorem 3.1, which will now be completed. (Some of the arguments that follow appear also in very similar form in [SV], so that the reader may consult there for more details.)

**Proof of theorem 3.1.** The lemma shows that for obtaining the stated estimates, only special diagrams need to be considered. (Since the 4-crossing diagram does not violate any of the two assertions of the theorem, let us exclude it in any further consideration.)

Assume that  $D$  has a Seifert circle of valence  $\geq 4$ . If we have a  $\tilde{r}_2$ -irreducible special diagram with a Seifert circle of valence  $\geq 4$ , then one can always perform a Reidemeister II move on any pair of non-neighbored edges in this Seifert circle region. Then one obtains a special diagram of the same Euler characteristic and two crossings more. By proper choice of the pair of edges, the new crossings can be made to form a separate new  $\sim$ -equivalence class, so that the diagram is still  $\tilde{r}_2$ -irreducible (although not always any pair of such edges will do).

This means that we need to consider only special diagrams  $D$  with Seifert circles of valence 2 and 3. In other words, the Seifert graph (see [Cr, §1])  $G$  of  $D$  is 2-3-valent (all its vertices have valence 2 or 3). Removing vertices of valence 2 in  $G$  by unbisections means identifying edges which correspond to  $\sim$ -equivalent crossings in  $D$ . Let  $G'$  be the 3-valent (cubic) graph obtained this way. Since  $D$  is reduced,  $G'$  is connected and contains no loop edges. Note that  $G$  is a reduced bisection of  $G'$ .

However,  $G'$  may be a single loop, which occurs when  $\chi(D) = 0$ . Then one obtains the  $(2, k)$ -torus link diagrams with reverse orientation (which are generated by the Hopf link and have one  $\sim$ -equivalence class). Thus we can exclude the degenerate case  $\chi(D) = 0$  in the rest of the proof.

Then, if  $e(G')$  is the number of edges of  $G'$  and  $v(G')$  the number of its vertices, we have  $v(G') = \frac{2}{3}e(G')$ , and

$$-\frac{e(G')}{3} = v(G') - e(G') =: \chi(G') = \chi(G) = \chi(D).$$

Since the number of  $\sim$ -equivalence classes in  $D$  is at most  $e(G')$ , we obtain the first result.

Now consider the second statement in the theorem. From Hirasawa's algorithm we know that the maximal crossing number of a generator is achieved by a special diagram (except for the case of knot diagrams of genus one), and that among such diagrams, in a diagram whose Seifert graph is 2-3-valent. Such a graph has  $2 - \chi(D)$  faces (corresponding to regions of  $D$  which do not contain a Seifert circle). For a while forget about the orientation of any component. If every  $\sim$ -equivalence class of such a diagram  $D$  had 2 crossings, the number of components of  $D$  would be equal to its number  $2 - \chi(D)$  of non-Seifert circle regions. Since the change of 2 crossings to 1 in each  $\sim$ -equivalence class

$$\text{X} \rightarrow \text{X} \quad (26)$$

changes the number of components at most by  $\pm 1$ , we need at least  $2 - \chi(D) - n(D)$  such replacements to obtain a diagram of  $n(D)$  components. Thus the maximal number of crossings we can have is

$$-6\chi(D) - (2 - \chi(D) - n(D)) = -5\chi(D) + n(D) - 2,$$

and we showed

$$c(D) \leq \begin{cases} -5\chi(D) + n(D) - 2 & \text{if } \chi(D) < 0 \\ 2 & \text{if } \chi(D) = 0 \end{cases} . \quad (27)$$

To obtain (23), we must show that we can improve the estimate in (27) by one crossing if  $\chi < 0$  and  $n < 2 - \chi$  (we excluded knot diagrams of genus one).

Assume there is a diagram  $D$  with  $-5\chi + n - 2$  crossings. Let  $G$  be its 2-3-valent Seifert graph, and  $G'$  be the 3-valent graph obtained from  $G$  by deleting valence-2 vertices. Call the edges of  $G'$  obtained under such un-bisections even, and the others odd.  $G'$  inherits a particular planar embedding from  $G$ , and hence we can build its dual graph  $G'^*$ . Note that each edge  $e'$  of  $G'$  corresponds bijectively to an edge  $e'^*$  in  $G'^*$ . Define for a subgraph  $\Gamma \subset G'$  the “dual” graph

$$\Gamma^* := \{e'^* : e' \in \Gamma\}.$$

Let

$$\Gamma := \{e' \in G' : e' \text{ odd}\}.$$

We claim that  $\Gamma^*$  is loopless. Assume that  $\Gamma^*$  contains a loop made up of edges  $l_1, \dots, l_k$ . Take the diagram  $D_0$  with the same graph  $G'$ , but such that all edges are even. Then  $D$  is obtained from  $D_0$  by  $2 - n - \chi$  moves (26), corresponding to the odd edges of  $G'$ . (The order of these moves is irrelevant.) To have  $n$  components in  $D$  (and not more), each such move must reduce the number of components by one, so that both strands on the left of (26) must belong to different components. However, if one performs on  $D_0$  the moves (26) corresponding to  $l_1, \dots, l_{k-1}$ , and then one applies the move for  $l_k$ , it is easy to see that both strands on the left of (26) for this last move belong to the same component. This contradiction shows that  $\Gamma^*$  is loopless.

Since  $\Gamma^*$  is loopless,  $\Gamma^* \subset \Gamma'^*$  for some spanning tree  $\Gamma'^*$  of  $G'^*$ . Then  $G' \setminus \Gamma'$  is also a spanning tree (of  $G'$ ). Thus  $G'$  has a spanning tree made up of even edges.

To show that this is impossible, consider the orientation of the Seifert circles corresponding to the vertices of  $G$ . Sign the vertices of  $G$  positive or negative depending on this orientation. (This makes  $G$  bipartite.) Then this signing reduces also to a signing of  $G'$ , with the property that even edges connect vertices of the same sign, and odd edges connect vertices of opposite sign. Since  $n < 2 - \chi$ , there must be vertices of  $G'$  of both signs. But  $G'$  has a spanning tree made up of even edges, and so all its vertices must have the same sign, a contradiction. (Clearly adding just one edge to  $\Gamma$  may solve the problem, because  $\Gamma^*$  may no longer be loopless, and  $G' \setminus \Gamma$  may get disconnected.)

This completes the proof of theorem 3.1.  $\square$

**Remark 3.1** Note that we always have  $2 - \chi(D) \geq n(D)$ , so that we can always eliminate  $n$  for  $\chi$ .

**Remark 3.2** In practice, it will be often convenient to use (27) rather than (23), because we can absorb by this additional crossing two of the exceptional cases (and avoid unpleasant case distinctions). Nonetheless, the additional argument for (23) is useful to settle completely the question on the maximal crossing numbers of generators in many cases, in particular for knots.

**Remark 3.3** It is clear from the proof that there is no natural way to obtain an upper restriction on the number of  $\sim$ -equivalence classes of odd number of crossings. In fact, no such restriction exists: in the case of knots and genus  $g \geq 6$  we constructed in [St2] generator diagrams with the maximal number of  $-3\chi = 6g - 3$   $\sim$ -equivalence classes, each one having a single crossing.

**Remark 3.4** If  $G'$  is 3-connected, then it is easy to see that in fact different edges of  $G'$  correspond to different  $\sim$ -equivalence classes in  $D$ , so that one easily finds examples where the first bound  $-3\chi$  in theorem 3.1 is realized sharply. We will study such generators  $D$  in more detail in a subsequent paper [St10].

### 3.4 Applications and improvements

We start with some first applications of the crossing number estimates for generators. For this we use the degree-2-Vassiliev invariant  $v_2 = \frac{1}{2}\Delta''(1)$ . Beside the independent interest of these inequalities, their proofs also introduce an idea that, in modified form, will be important for the later more general results.

**Theorem 3.2** If  $K$  is a positive knot, then each reduced positive diagram  $D$  of  $K$  has  $c(D) \leq 9g(K) - 8 + 2v_2(K)$  crossings, except if  $K$  is the trefoil (where  $c(D) \leq 4$ ).

**Proof.** Let  $D'$  be the positive generator in whose series  $D$  lies. We have  $g(D') = g(D) = g(K)$ , and we know that a positive  $\tilde{t}_2$ -twist in a positive diagram augments  $v_2$ . If  $g(D') > 1$ , then  $c(D') \leq 10g(D') - 7$  by corollary 3.1. On the opposite side, we have  $c(D') \geq 2g(D') + 1$ , so that by theorem 6.1 of [St]

$$v_2(D') \geq \frac{c(D')}{4} \geq \frac{g(D') + 1}{2}$$

(where in the second inequality we used the obvious integrality of  $v_2$ ). Now we can apply at most  $v(D) - v_2(D')$   $\tilde{t}_2$ -twists to  $D'$  to obtain  $D$ , so that

$$c(D) \leq c(D') + 2(v_2(K) - v_2(D')) \leq 10g(D') - 7 + 2v_2(K) - g(D') - 1 = 9g(K) - 8 + 2v_2(K).$$

If  $g(D) = g(D') = 1$ , then we must show  $c(D) \leq 2v_2(K) + 1$ . This can be checked directly, since by theorems 2.11 and 2.12,  $D$  is either a  $(p, q, r)$ -pretzel diagram for  $p, q, r \geq 1$  odd, or a positive rational knot diagram  $C(p, -q)$ , with  $p, q \geq 2$  even. In former case  $c(D) = p + q + r$  and  $v_2(D) = (pq + pr + qr + 1)/4$ , and the inequality easily follows. In latter case  $c(D) = p + q$  and  $v_2(D) = pq/4$ , so the inequality holds unless  $p = q = 2$ , which is the positive 4-crossing trefoil diagram.  $\square$

**Theorem 3.3** If  $K$  is an almost positive knot, then each reduced almost positive diagram  $D$  of  $K$  has  $c(D) \leq 10g(K) + 3 + 2v_2(K)$  crossings.

**Proof.**  $D$  has genus at most  $g(K) + 1$  by theorem 2.3. Consider the almost positive generator  $D'$ , from which  $D$  is obtained by positive  $\tilde{t}_2$ -twists. Since almost positive genus 1 diagrams belong to positive knots,  $g(D') > 1$ , so that by corollary 3.1,  $D'$  has at most  $10g(D') - 7 \leq 10g(K) + 3$  crossings. Moreover, since  $g(D') > 1$ ,  $D'$  is not an unknotted twist knot diagram. Thus  $v_2(D') > 0$  by theorem 4.1 of [St3].

Now we apply in a proper order the  $\tilde{t}_2$ -twists taking  $D'$  to  $D$ , keeping track of  $v_2$ . We know from the Polyak-Viro formula for  $v_2$  that a  $\tilde{t}_2$ -move augments  $v_2$  except if it is at a crossing  $p$ , linked with only two other crossings,  $q$  and  $r$ , one of which, say  $q$ , is the negative one. (In this situation  $v_2$  is preserved.) Call such a crossing  $p$  *thin*. In the proof of lemma 5.3 of [St3] we observed that then  $q$  and  $r$  form a trivial clasp in  $D'$ . To avoid the elimination of  $q$  in  $D$ , this clasp must be parallel, and a twist at  $r$  must be applied in the sequence of  $\tilde{t}_2$ -twists taking  $D'$  to  $D$ . Now we count how many different crossings occur as  $r$  in the above situation (for some  $p$ ). Since  $q \approx r$ , and  $\approx$  is an equivalence relation, any two possible  $r$  and  $r'$  are  $\approx$ -equivalent. Then they intersect the same set of other chords, in particular  $p$ . But  $p$  intersects at most one other chord different from  $q$ . So no two different  $r$  exist, although for the unique  $r$  there may be different  $p$ . However, applying first the  $\tilde{t}_2$ -twist at  $r$ , we eliminate all these crossings  $p$  as thin crossings (and there are no other ones). So we ensure that all following twists augment  $v_2$ . Since  $v_2(D') > 0$ , at most  $v_2(K) = v_2(D)$  twists can be performed (the initial one included). Thus

$$c(D) \leq c(D') + 2v_2(K) \leq 10g(K) + 3 + 2v_2(K). \quad \square$$

Even albeit the inequalities in theorem 3.1 are optimal in general, they can be improved under additional conditions. One such condition is related to the *signature*  $\sigma(L)$  of the underlying link  $L$ , in comparison to the maximal *Euler characteristic*  $\chi(L)$  of an orientable spanning surface of  $L$ . It is well-known [Mu3] that  $|\sigma(L)| \leq 1 - \chi(L)$ , at least when  $L$  does not bound disconnected surfaces of small genus, so for example for knots or (non-split) positive links. We can say something about the situation when  $\sigma(L) \ll 1 - \chi(L)$ .

**Lemma 3.2** Assume we have a diagram  $D$  of a link  $L$  with  $\chi(D) < 0$  and  $a(D)$  negative crossings, and

$$k(D) := \frac{1 - \chi(D) - 2a(D) - \sigma(D)}{2} \geq 0.$$

Then  $D$  has at most  $-3\chi(D) - \frac{3}{2}k(D)$   $\sim$ -equivalence classes, and its underlying generating diagram has at most

$$-5\chi(D) + n(D) - 2 - \frac{1}{2}k(D)$$

crossings.

**Proof.** If  $D$  is a 4-crossing knot diagram, then  $k(D) = 0$ , so this case follows easily. Otherwise, let  $\tilde{D}$  be the special diagram obtained from  $D$  by Hirasawa's algorithm. Hereby, when applying a move at a separating Seifert circle  $s$ , the new strand is placed so as to create  $\text{ind}(s) - 1$  (and not  $\text{ind}(s)$ ) new negative (non-nugatory) crossings.

We assume first that  $\text{ind}(s) \geq 3$ . We have shown in the proof of theorem 3.1 that the number of  $\sim$ -equivalence classes is augmented by

$$2\text{ind}(s) - 3 \geq \frac{3}{2}(\text{ind}(s) - 1), \quad (28)$$

and that of the crossing number of the underlying generating diagram at least by

$$2\text{ind}(s) - 5 \geq \frac{1}{2}(\text{ind}(s) - 1). \quad (29)$$

As for the positive special alternating diagram  $\hat{D}$  obtained from  $\tilde{D}$  by crossing changes, we have from [Mu3]  $\sigma(\hat{D}) = 1 - \chi(\hat{D}) = 1 - \chi(\tilde{D})$ , and  $\sigma$  changes at most by 2 under a crossing change, we have that  $\hat{D}$  has at least

$$\frac{1 - \chi(D) - \sigma(D)}{2} \quad (30)$$

negative crossings. Thus  $\hat{D}$  has at least  $k(D)$  negative crossings added by the Hirasawa moves (and not inherited from  $D$ ). Then apply (28), (29) and theorem 3.1.

This argument shows the assertion if no Hirasawa moves at index-2 Seifert circles are applied. To deal with  $\text{ind}(s) = 2$  we consider again the modified Hirasawa move (25). Then we refine our argument as follows. Mark in each modified Hirasawa move the negative crossing in the additional clasp (this is crossing  $a$  in (25)). Then we claim that  $\tilde{D}$  has (30) many (not only negative but) non-marked negative crossings. To see this, it suffices to show that the switch of all marked (positive) crossings in  $\tilde{D}$  preserves  $\sigma$ . Now, switching all marked crossings in  $\tilde{D}$  and resolving the trivial clasps gives a diagram  $\hat{\tilde{D}}$ . Clearly  $\hat{\tilde{D}}$  is special and positive. But it has also the same  $\chi$  as  $\tilde{D}$ , since modulo crossing changes it is obtained from  $D$  by the ordinary (non-modified for  $\text{ind}(s) = 2$ ) Hirasawa moves. Thus  $\sigma(\hat{\tilde{D}}) = \sigma(\tilde{D})$  follows from Murasugi's result  $\sigma = 1 - \chi$ .

Therefore, (25) creates (with proper choice between putting the arc above or below) only one non-marked negative crossing, but at least two new  $\sim$ -equivalence classes, and augments the generator crossing number at least by one. Then (28) resp. (29) remain valid with the l.h.s. being the augmentation of  $\sim$ -equivalence classes and generator crossing number resp., and the parenthetical term on the r.h.s. the number of (non-marked) negative crossings contributing to (30).  $\square$

**Remark 3.5** Observe that, for knots, we can use instead of  $\sigma$  the Ozsváth-Szabó signature  $\tau$ , since it coincides (up to a factor) with  $\sigma$  on (special) alternating knots, and enjoys (up to that factor) the property (12). However, Livingston in fact deduced the Rudolph-Bennequin inequality (2) (in the original form of the estimate, without the term  $s_-(D)$  on the right) from  $\tau$ , and so lemma 3.2 becomes trivial for  $\tau$ . A similar comment applies on the signature  $s$  derived from Khovanov homology (see Bar-Natan [BN] for example).

Note that the signature improvement of theorem 3.1 can be applied to theorems 3.2 and 3.3. However, we see in [St8] that, with further tools introduced, the variety of possible modifications and improvements of such inequalities grows considerably, so that we cannot discuss each one in detail. We will thus in particular not elaborate much on the use of  $\tau$  and  $s$ , and leave the adaptation to an interested reader.



## 4 Generators of genus 4

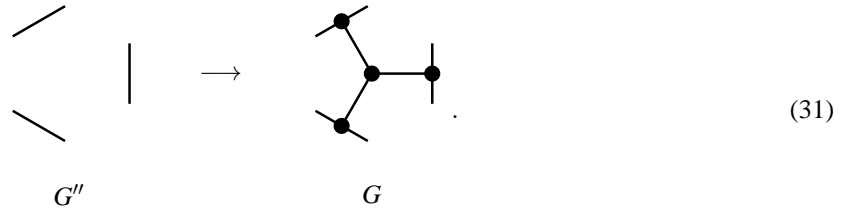
The methods developed in §3 allow us to push the compilation of generators for genus 2 and 3 in [St2] one step further, and to settle genus 4. Beside its algorithmical merit, this project had the purpose of verifying (and in fact correcting minor mistakes in) the theoretical thoughts, on which our approach bases. Several applications presented below in this paper rely on this compilation.

Table 1 gives the distribution of the prime (knot) generators  $K$  of genus 4 by crossing number and number of  $\sim$ -equivalence classes. For maximal generators, we know from our previous work (theorem 3.1) that the crossing numbers in question lie between 21 and 33, but excluded 21 in [St2] by a computational argument using the property that the Seifert graph must be (3-valent and) bipartite. For 22 crossings, still there are no maximal generators.

**Theorem 4.1** There are 3,414,819 prime knot generators of genus 4. Among them there are 1,480,238 special and 1,934,581 non-special ones.

Note that the effort with every new genus increases dramatically, and new theoretical insight was needed to reduce calculations to a feasible magnitude. In our proof we will explain the algorithm used now. (Still it seems out of scope to attempt the case  $g = 5$ , already for reasons of storing and maintaining the result.)

**Proof.** For the maximal generators we use the work in [SV]. We have to determine first the planar 3-connected graphs  $G$  having knot markings. They can be obtained by a (graphic version of the)  $\gamma$ -construction in [V] from graphs  $G''$  of genus 3:



It is easy to see that if the graph  $G$  on the right of (31) is 3-connected, then the graph  $G''$  on the left must be at least 2-connected. We determined the graphs  $G''$  from the list of maximal Wicks forms of genus 3 compiled by A. Vdovina. (Note that in the words an edge is a pair of inverse letters  $a^{\pm 1}$ , while a vertex is a set of 3 letters  $\{a, b, c\}$ , such that, for some choice of signs,  $a^{\pm 1}b^{\pm 1}$ ,  $b^{\pm 1}c^{\pm 1}$  and  $c^{\pm 1}a^{\pm 1}$  are subwords.) From the graphs thus found, we selected only the planar and 2-connected ones, and obtained 34 graphs  $G''$ .

Then the graphs  $G$  on the right of (31) are generated by a  $\gamma$ -construction. In applying the move (31), some restrictions can be taken into account. In order  $G$  to become 3-connected, on the left of (31) not all 3 segments can belong to the same edge in  $G''$  (although two of them can). Moreover, whenever (a segment of) one in a copy of a multiple (hence double) edge in  $G''$  is affected, the other copy cannot be. Otherwise, the graph  $G$  is non-planar or still only 2-connected, or  $G''$  is not 2-connected.

All  $G$  obtained were tested for planarity and 3-connectedness (and, of course, isomorphy). We obtained 50 graphs  $G$ . This list was later verified by the program of Brinkmann and McKay [BMc] (which I discovered only afterwards and which made it possible to deal with maximal generators of genus 5 and 6; see [St10]).

Then for each graph  $G$  on the right of (31), we selected all vertex orientations with the orientation of one vertex fixed, and bisected edges between equally oriented vertices. The new graph  $G'$  corresponds to a knot iff the number of its spanning trees is odd (see [MS]).

All such graphs  $G'$  were again tested for isomorphy. We proved in [SV] that the alternating diagrams with these graphs as Seifert graphs do not admit flypes, so that there is a one-to-one correspondence between alternating knot, alternating diagram (up to mirroring and orientation) and Seifert graph. Note that isomorphy between the  $G'$  is preserved under unbisecting edges, so that isomorphic  $G'$  must come from the same  $G$ , and the isomorphy check for  $G'$  can be done for each  $G$  separately. After duplications of  $G'$  were eliminated, the Dowker notations [DT] were generated, and so 42,294 maximal generators obtained (which were later confirmed by the calculation in [St10]).

**Table 1:** The number of  $\tilde{r}_2'$  irreducible prime genus 4 alternating knots tabulated by crossing number  $c$  and number of  $\sim$ -equivalence classes ( $\# \sim$ ).

<div><div>c</div><div># ~</div></div>	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	total
8								29																		29
9	1	2	10	28	71	104	147			145																508
10		21	72	210	356	557	660	819	1092			369														4156
11			48	257	766	1791	2942	3832	3080	2804	3188			447												19155
12				55	487	2033	4734	8585	12145	13523	8500	5168	4707			313										60250
13					56	548	3087	9661	19112	27552	31293	27717	14629	5427	3876			111								143069
14						46	590	3519	13251	32388	52870	61747	53398	35540	15787	3173	1827			20						274156
15							41	489	3584	14749	41049	78373	102880	95709	61646	28311	10626	965	465			1				438888
16								27	356	2814	13781	42566	90877	135278	138221	100392	48096	13094	4195	115	49					589861
17									14	231	1854	9704	34955	83859	141210	163710	125842	68515	24978	2942	837					658651
18										4	96	989	5258	20307	56939	110240	150023	136642	75688	27646	8127	172	46			592177
19											4	25	300	2109	8414	25220	57598	92985	105424	77316	29771	5059	1302			405527
20													6	52	401	2181	6905	17039	32977	45891	44939	27879	7828			186098
21															9	36	205	876	2328	4882	8272	10236	9024	5094	1332	42294
total	1	23	130	550	1736	5079	12201	26961	52634	94210	152635	226658	307010	378728	426503	433576	401122	330227	246055	158812	91995	43347	18200	5094	1332	3414819

We also checked knot-theoretically that these generators are distinct. The skein polynomial distinguished all except 70 pairs. Among these pairs, the Kauffman polynomial dealt with 62. The remaining 8 pairs were distinguished by the hyperbolic volume.

We know from [SV] that one can obtain the other, not maximal, *special* generators from the maximal ones by resolving reverse clasps. Then for each diagram of a maximal generator we resolved any possible set of (reverse) clasps, and identified among all these (alternating) knots the generators of genus 4. They are 1,480,238.

Then we generated all possible special diagrams by flypes. From these diagrams one can then obtain the other (non-special) generating diagrams inductively by the number of separating Seifert circles, using the (reverse to) the moves of Hirasawa (figure 10), and the study of the effect of these moves on generators, carried out in §3.

We start with a generator diagram  $D$  and want to determine all generator diagrams  $D'$  obtained from  $D$  by undoing a Hirasawa move. This means that we must identify the segment  $S$  in  $D$  created by the move. We call  $S$  below the *Hirasawa segment*. There are several properties of  $S$  which follow from our treatise.

1. This is a segment of odd length (i.e., odd number of crossings on it), and no self-intersection.
2. Let  $x$  and  $y$  be the first and last crossing on  $S$ , and call the other crossings *internal*. By the work in §3, we know that no internal crossings should lie in a non-trivial  $\sim$ -equivalence class.
3. Moreover,  $x$  and  $y$  have at most 2  $\sim$ -equivalent crossings each. So, prior to undoing the Hirasawa move, we have the option of creating (by a  $\tilde{I}_2$ -move and possible flypes) two crossings  $\sim$ -equivalent to  $x$  and/or  $y$ , if  $x$  and/or  $y$  is in a trivial  $\sim$ -equivalence class.
4. After the optional application of such  $\tilde{I}_2$ -moves and flypes, we have the additional condition that the length of the Hirasawa segment  $S$  is strictly less than the number of crossings outside  $S$ .
5. All possible such segments  $S$  were generated, and their crossings removed. Then the resulting Gauß codes of  $D'$  were tested for realizability,  $\tilde{I}_2$ -reducedness, primeness and for having the correct genus (4) and number of separating Seifert circles (one more than in  $D$ ).

Note that we tested only necessary conditions for  $S$ . So some of the diagrams  $D'$  we obtain this way may actually not give  $D$  by a Hirasawa move. The difficulty is, though, to obtain all diagrams  $D'$  we need at least once, and we accomplish this, since we reverse all correct Hirasawa moves. The decisive merit of the Hirasawa move is that the way it provides to find all the  $D'$  is much more economical, in comparison to the previous algorithm in [STV, St2]. Duplicated diagrams  $D'$  can be subsequently eliminated by bringing the Dowker notation of any diagram to standard form.

One can then iterate this procedure, undoing (potential) Hirasawa moves (and performing the necessary consistency checks), without the need to flype in between. The process terminates, when for some number of separating Seifert circles (8 here, and in general  $2g$  by an easy argument), no more (regular) diagrams arise. The flypes are needed after the generation of the diagram lists to identify the underlying knots, and the work is finished.  $\square$

In order to secure that this algorithm is practically correct (theoretically it is justified by the proof of theorem 3.1), we verified that the subsets of generators we obtained for  $\leq 18$  crossings coincide with those that can be selected from the alternating knot tables in [HT] and [FR]. Prior to starting genus 4, we also applied this procedure for genus 3. We checked that the sets of generators we obtain coincide with those generated previously in [St2], by the algorithm described there and in [STV]. While the new implementation dealt with genus 3 about 50 times faster, it was still not before several weeks of labour and computation (partly on distributed machines) that table 1 became complete.

A first small application is easy to obtain, and useful to mention.

**Corollary 4.1** There are 680 achiral generators of genus 4. The number of prime achiral alternating genus 4 knots of crossing number  $n$  grows like  $\frac{43}{2^8 8!} n^8 + O(n^7)$  for  $n \rightarrow \infty$  even.

**Proof.** As in [St2], one must find the achiral generators. The use of Gauß sums (see proposition 13.3 of [St2]) pre-selects 798 generators, and 680 of them are indeed found achiral by the hyperbolic symmetry program of J. Weeks. The check of the maximal number of  $\sim$ -equivalence classes of such generators shows it to be 18. This number is attained by 43 of the achiral knots, and none of them has non-trivial  $\approx$ -equivalence classes, or admits (even trivial) type A flypes (as explained in figure 5).  $\square$

## 5 Unknot diagrams, non-trivial polynomials and achiral knots

### 5.1 Some preparations and special cases

We apply now the classification of genus 4 generators to prove theorem 1.1. We begin with an easy proof for 2-almost positive diagrams, that outlines the later methods.

**Proposition 5.1** If  $D$  is a 2-almost special alternating diagram of the unknot, then  $D$  has a trivial clasp.

**Proof.** We use the Bennequin inequality stated in theorem 2.3. It says that  $g(D) \leq 2$ .

If  $D$  is composite or of genus 1, then the claim is easy to obtain using the genus 1 case [St3]. So assume  $g(D) = 2$  and  $D$  is prime.

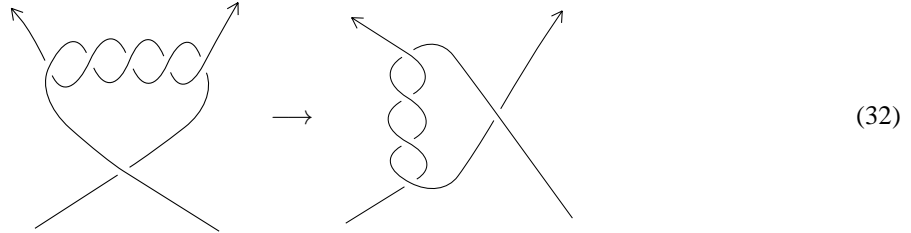
If we have a  $\sim$ -equivalence class which has crossings of opposite (skein) sign, we either have a trivial clasp, or if not, we have a non-trivial  $\sim$ -equivalence class which admits flypes. If after the flype and resolution of the clasp we still have a diagram of genus two, we see that we did not start with a diagram of the unknot. By a direct check of the special genus two generators, we verify that this always happens.

We can thus assume that no  $\sim$ -equivalence class has crossings of opposite (skein) sign. We call a  $\sim$ -equivalence class positive or negative after the sign of its crossings.

Let  $\hat{D}$  be the 2-almost positive (generator) diagram, obtained when removing in  $D$  (by undoing  $\vec{t}_2^2$  twists) crossings in (positive)  $\sim$ -equivalence classes, so that at most 2 crossings in each class remain. Then  $\sigma(\hat{D}) \leq 0$ .

Assume  $\hat{D}$  has two twist equivalent crossings  $x$  and  $y$  of different sign. A reverse clasp in  $\hat{D}$  persists under  $\vec{t}_2^2$  twists, even up to flypes, so  $x$  and  $y$  must be  $\approx$ -equivalent. Let us call their equivalence class *bad*.

One can flype the negative crossing  $x \in \hat{D}$  to form a parallel clasp with  $y$ , and since  $x$  is not twisted at in  $\hat{D}$  when recovering  $D$ , this flype persists in  $D$ . Possible  $\vec{t}_2^2$  twists in  $D$  at  $y$  create a  $(2k+1, -1)$ -rational tangle. Then by a  $(2k+1, -1) \rightarrow (2k, 1)$  rational tangle move in  $D$  (which is a particular type of wave move),



we find a diagram  $D_1$  with one crossing less and  $\sigma \leq 0$ , so that  $c(\hat{D}_1) = c(\hat{D}) + 1$ , and  $\hat{D}_1$  has one bad class less than  $\hat{D}$ . Still  $D_1$  is  $\leq 2$ -almost positive, though not necessarily special. Repeat now such a move until  $\hat{D}_1$  has no bad class.

Now by direct check all 2-almost positive genus 2 generator diagrams  $\hat{D}$  with  $\sigma \leq 0$  and no bad class have  $\leq 7$  crossings and are not special. (Actually  $\sigma = 0$ , since  $\sigma = 4$  on positive genus two diagrams.) They occur for the generators  $6_3$ ,  $7_6$  and  $7_7$ .

Since  $D_1$  is not special,  $D_1 \neq D$ , and  $7 \geq c(\hat{D}_1) > c(\hat{D})$ , so  $c(\hat{D}) \leq 6$ . The only special generator coming in question for  $\hat{D}$  is  $5_1$ . However, it is easy to check that 2-almost positive 5-pretzel diagrams of the unknot have a clasp, which completes the proof.  $\square$


**Proposition 5.2** If  $D$  is a 2-almost positive diagram of the unknot, then  $D$  can be made trivial by Reidemester moves not augmenting the crossing number.

**Proof.** If  $D$  is special, of genus 1 or composite, the claim follows from the almost positive case and the previous proposition. So we assume that  $D$  is of genus two, prime and non-special.

We take all non-special genus 2 generators, apply all possible flypes, and switch crossings so that the diagrams are 2-almost positive. We discard all diagrams of  $\sigma > 0$ , and from those of  $\sigma = 0$  remove the duplicates.

If now we have crossings of opposite sign forming a (trivial) reverse clasp, this clasp will remain after flypes, and the diagram is simplifiable using a Reidemeister II move. We discard thus diagrams with trivial reverse clasps.

If we have a trivial parallel clasp, we need to have a  $\frac{1}{2}$ -twist at the positive crossing. We apply this twist in each such clasp, and discard all diagrams of  $\sigma > 0$ . There remain only 3 diagrams in the series of  $6_2$ . They contain a

tangle  with a reverse clasp. This tangle can be simplified by Reidemeister III unless there is a twist at the crossings in the reverse clasp. Then again we check that  $\sigma > 0$ . □

We prepare some arguments that will be useful in the proofs of theorems 1.1, 1.2 and 1.3.

**Lemma 5.1** Assume  $D$  is a  $k$ -almost positive diagram of genus  $k$ , and  $D$  depicts an achiral knot. Then  $P(D) = 1$ .

**Proof.** By Morton's inequality (16), we have  $\min \deg_l P(D) \geq 0$ . So achirality implies that  $P(D) \in \mathbb{Z}[l^2]$ , i.e. it has only a non-trivial  $m$ -coefficient in degree 0. The [LM, proposition 21] implies that  $P(D) = 1$ . □

**Lemma 5.2** Assume  $D$  is a  $k$ -almost positive diagram of genus  $k$ , and  $P(D) = 1$ . Then for each pair  $(a, b)$  of Seifert circles in  $D$ , connected by a negative crossing, there are at least two crossings connecting  $(a, b)$ .

Moreover, if  $k \leq 4$ , then for at least one pair  $(a, b)$  of Seifert circles connected by negative crossings, there is a positive crossing connecting  $(a, b)$ .

Using lemma 2.2, we have

**Corollary 5.1** Any  $k$ -almost special alternating diagram with  $P = 1$  of genus  $k \leq 4$  contains a trivial clasp up to flypes. □

This immediately discards the special genus 4 generators.

**Proof of lemma 5.2.** We use (18). Since Morton's inequality is sharp, we see that  $\text{ind}_-(D) = 0$ , so that each negative crossing has a Seifert equivalent one. If no negative crossing has a Seifert equivalent positive one, then for  $k \leq 4$ , the negative crossings lie in at most 2 Seifert equivalence classes. Then an easy observation shows that the diagram  $D$  is  $A$ -adequate. So by theorem 2.7,  $V(D) \neq 1$ , and then the same is true for  $P(D)$ . □

## 5.2 Reduction of unknot diagrams

We prove here first a slightly weaker version of theorem 1.1, which allows for flypes. We will explain in the next subsection how to remove the need of flypes.

**Proposition 5.3** For  $k \leq 4$ , all  $k$ -almost positive unknot diagrams are trivializable by crossing number reducing wave moves and flypes.

**Proof.** We start with a  $\leq 4$ -almost positive unknot diagram  $D$ , and will show that one can perform, after possible flypes, a wave move on  $D$ . In the following a wave move is always understood to be one that *strictly reduces* the crossing number  $c(D)$  of  $D$ . Then we work by induction on  $c(D)$ . Note that a wave move does not augment neither

the number of positive, nor the number of negative crossings (while it reduces one of both). So it does not lead out of the class of  $\leq 4$ -almost positive diagrams. It is possible that the diagram genus becomes larger than 4. However, in this case we know that  $D$  cannot be a diagram of the unknot. So we need to be just interested in either showing that  $D$  is not an unknot diagram, or finding a wave move.

Next, genus and number of negative crossings is additive under connected sum of diagrams. It is then sufficient to work with prime diagrams  $D$ , with the following additional remark. If we have a composite diagram  $D = A \# B$ , then flypes in  $A$  and  $B$  are not affected by taking the connected sum. However, we may have a bridge  $a$  in a factor diagram  $A$ , which is spoiled when taking the connected sum with  $B$ . This can be remedied by displacing  $B$  out of  $a$ . Such a move can formally be regarded as a flype, if in figure 4 one of  $P$  or  $Q$  has a diagram in which one of its strands is an isolated trivial arc. (Let us call this move *factor sliding flype*; it is as in figure 2, except that the tangle is additionally flipped.)

Let  $D$  be the initial diagram. Clearly  $g(D) \leq 4$ . Now let  $\hat{D}$  be  $D$  with *positive*  $\sim$ -equivalence classes reduced (by  $\vec{l}_2$  moves) to 1 or 2 crossings. (That is, if a negative  $\sim$ -equivalence class has  $> 2$  crossings,  $\hat{D}$  is not a crossing-switched generator diagram.) Clearly  $\sigma(\hat{D}) \leq 0$ .

Let  $k$  be the number of negative  $\sim$ -equivalence classes in  $D$  (or equivalently  $\hat{D}$ ), and  $\alpha = (a_1, \dots, a_k)$  be the vector of their sizes (number of crossings). We sort the integers  $a_i$  non-increasingly, ignoring their order.

Let  $D'$  be the positification of a diagram of a generator, whose series contains  $D$  and  $\hat{D}$ . That is,  $D'$  is obtained from  $D$  or  $\hat{D}$  by replacing each  $\sim$ -equivalence class of  $n$  crossings with a class of one positive crossing if  $n$  is odd and 2 positive crossings if  $n$  is even. Then  $\sigma(D') \leq 2k$ , since twists in a  $\sim$ -equivalence class do not alter  $\sigma$  by more than  $\pm 2$ . In particular, we can exclude the case  $k = 1$  (i.e.  $\alpha$  being one of (4), (3) or (2)), because for a positive diagram  $D'$  the property  $\sigma = 2$  implies (see end of §2.5) that  $g(D') = g(D) = 1$ , and then we are easily done by [St4] identifying genus 1 unknot diagrams.

In the first step to generate  $D$ , we try to obtain  $\hat{D}$  from  $D'$ , while  $D'$  is obtained directly from the generator table by flypes and positification. Flypes on the generator are necessary, because they commute with the  $\vec{l}_2$ -twists only up to mutations, which we try to avoid (in our stated repertoire of moves simplifying the unknot diagram).

For the possible  $\alpha$ , the potential diagrams  $\hat{D}$  are determined by crossing switches (and in the case of  $\alpha = (3, 1)$ , one  $\vec{l}_2$  twist).

It is convenient to check the bound for  $\sigma$ , which is a necessary condition for  $\sigma(\hat{D}) \leq 0$ , after each crossing switch. Moreover, for given  $\alpha$  we select only the  $D'$  with  $\sigma(D') \leq 2k$ . This allows to discard a lot of possible continuations of the crossing switch procedure, which are understood not to lead to a proper  $\hat{D}$ . Also, one should order the  $\sim$ -equivalence classes of 1 and 2 elements and switch w.r.t. this order to avoid repetitions. Concretely, we proceeded as follows.

- $\alpha = (1, 1, 1, 1)$ . Change successively one crossing and check  $\sigma \leq 6$ ,  $\sigma \leq 4$ ,  $\sigma \leq 2$ ,  $\sigma \leq 0$ .
- $\alpha = (2, 1, 1)$ . Change a reverse clasp and two crossings and check  $\sigma \leq 4$ ,  $\sigma \leq 2$ ,  $\sigma \leq 0$ .
- $\alpha = (2, 2)$ . Change two reverse clasps and check  $\sigma \leq 2$ ,  $\sigma \leq 0$ .
- $\alpha = (3, 1)$ . Change twice one crossing and check  $\sigma \leq 4$  and  $\sigma \leq 2$ . Then perform a  $\vec{l}_2$  at one of the negative crossings and test  $\sigma \leq 0$ .
- The cases  $\alpha = (2, 1)$  and  $(1, 1, 1)$  are analogous.

The diagrams  $\hat{D}$  found are relatively few in comparison to the number of generators  $D'$ . Also,  $\sigma$  is a mutation invariant. Therefore, to handle the flypes in practice, it is easier to start with one diagram  $D'$  per generator, determine whether it gives a  $\hat{D}$ , and only then to process the other diagrams obtained by flypes from  $D'$ . Actually, one can apply the flypes in  $\hat{D}$  directly, except for  $\alpha = (3, 1)$ . Latter is easy to handle, since the  $D'$  with  $g(D') = 2$  we need to consider are few. Thus let us exclude this in the following.

If  $\hat{D}$  has a trivial clasp up to flypes, then all  $D$  arising from this  $\hat{D}$  will admit a  $(2k+1, -1) \rightarrow (2k, 1)$  rational tangle move (32) up to flypes, and so we can discard such  $\hat{D}$ . Note that only a negative crossing in a trivial  $\sim$ -equivalence

class can form a parallel clasp up to flypes, and since in passing from  $\hat{D}$  to  $D$  we do not twist at negative crossings, one can flype the negative crossing properly also in  $D$  for (32) to apply.

Let  $\hat{\mathcal{D}}$  be the set of diagrams  $\hat{D}$  obtained/maintained so far. In trying to reconstruct  $D$  from  $\hat{D}$  we distinguish at which positive  $\sim$ -equivalence classes we apply twists.

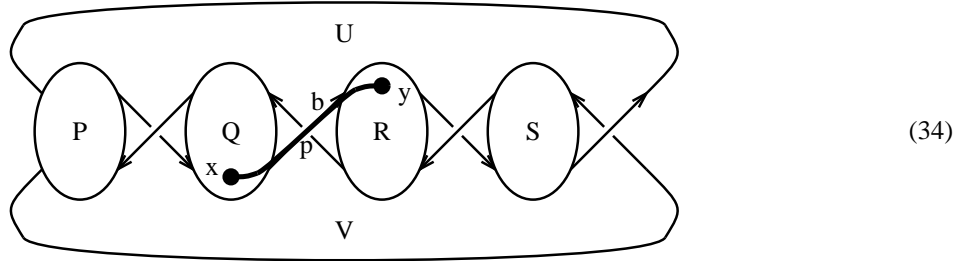
Let  $X$  be a diagram and  $\mathcal{X}$  a set of diagrams. Fix for  $X$  the set  $C_X$ , consisting of one crossing in each positive  $\sim$ -equivalence class in  $X$ . Let

$$X_* = \left\{ X' : \begin{array}{l} X' \text{ is obtained from } X \text{ by 0 or 1} \\ \text{twist at each crossing in } C_X \end{array} \right\}, \quad \text{and} \quad \mathcal{X}_* = \bigcup_{X \in \mathcal{X}} X_*. \quad (33)$$

(The choice to twist or not is made independently for each crossing, so  $|\mathcal{X}_*| = 2^{|C_X|}$ .) The reason we will make use of this construction shortly is the following

**Lemma 5.3** If a diagram  $D$  admits a wave move, and  $D$  has a  $\sim$ -equivalence class of at least 4 crossings, then one can flype and reduce  $D$  by a  $\vec{t}_2$ -move, so that the resulting diagram  $D'$  still admits a wave move. Contrarily, if a diagram  $D'$  admits a wave move, and  $D$  is obtained from  $D'$  by a  $\vec{t}_2$ -move at a non-trivial  $\sim$ -equivalence class, then  $D'$  admits a wave move after flypes.

**Proof.** We focus on the first claim. Let  $D$  be presented w.r.t. four of the crossings in the  $\sim$ -equivalence class  $T$ ,



with the (possibly empty) tangles  $P, Q, R, S$  being enclosed by these crossings. Let  $b$  be a bridge in  $D$  that can be shortened by a wave move, and  $\text{len } b$  the number of overcrossings  $b$  contains.  $b$  is understood to start and end on points  $x$  and  $y$  on the segment (edge) of the diagram line after/before the beginning/ending undercrossing. Now the existence of a wave move means that, if we delete  $b$  from the plane curve of  $D$ , the distance between the regions  $X$  and  $Y$  that contain the endpoints  $x, y$  of the resulting arc is smaller than  $\text{len } b$ . Distance between regions  $X$  and  $Y$  is meant as the shortest length of a sequence of successively neighbored regions, starting with  $X$  and ending with  $Y$ , counting *one* of  $X$  and  $Y$ , and neighbored regions means regions sharing an edge.

It is an easy observation that  $b$  contains at most one crossing in  $T$  (assuming these crossings have all the same sign). Let this be the crossing  $p$  between  $Q$  and  $R$ . (The case that  $b$  does not contain any crossing in  $T$  is dealt with a similar, though slightly simpler argument.) Now let  $\gamma = (X, \dots, Y)$  be a shortest length sequence of neighbored regions connecting  $X$  and  $Y$ . If  $\gamma$  contains a region  $W$  *completely* inside  $P$  or  $S$ , then it must contain also  $U$  and  $V$ . However, the distance of  $U$  and  $V$  is two, and moving from  $U$  to  $V$ , it is not necessary to pass via  $W$ . So  $\gamma$  can be chosen not to contain any regions completely inside  $P$  or  $S$  (though it may have to contain a region whose corner is one of the crossings in  $T$ ).

This means now that the flype that makes  $P$  and  $S$  empty (moving their interior into  $Q$  or  $R$ ), does not change the distance between  $X$  and  $Y$ . Likewise, it does not affect the bridge  $b$ . So  $b$  remains too long after such flypes. Now when  $P$  and  $S$  are empty, one can reduce  $D$  by a  $\vec{t}_2$  move, and so we are done with the first statement.

The proof of the second statement is similar. Again a wave move in  $D'$  can be chosen so that it does not affect the diagram outside  $Q, R$  and  $p$ . Then in  $D$  one can adjust by flypes the new crossings created by the  $\vec{t}_2$ -move at a crossing  $q$  to lie outside that portion of the diagram if  $p = q$ , and entirely within  $R$  or  $Q$  if  $p \neq q$ .  $\square$

Now from our set  $\hat{\mathcal{D}}$  we generate the set  $\mathcal{D} = (\hat{\mathcal{D}})_*$ . Our understanding is that one can obtain all  $D$  from an  $X \in \mathcal{D}$  by  $\tilde{r}_2'$  moves at a non-reduced  $\sim$ -equivalence class. The point in this restriction comes from the previous lemma, which says that the applicability of a wave move up to flypes persists under such  $\tilde{r}_2'$  moves. So if some  $X \in \mathcal{D}$  admits a wave move, one can discard all  $D$  obtained from  $X$  by twising at a non-reduced  $\sim$ -equivalence class. (In fact, the lemma shows that it is necessary to twist only at a crossing in a trivial positive  $\sim$ -equivalence class, but we will need (33) it is given form shortly below.) Similarly if  $\sigma(X) > 0$ , the same holds for all  $D$  obtained from  $X$  by positive  $\tilde{r}_2'$  twists. Such  $X$  can be likewise discarded.

Therefore, discard all diagrams  $X$  in  $\mathcal{D}$  admitting a wave move or having a positive signature, obtaining a subset  $\mathcal{E}$  of  $\mathcal{D}$ . We deal with  $\mathcal{E}$  using the maximal degree of the Jones polynomial. This procedure was described in [St14]. What we need in our case is the following property.

**Lemma 5.4** (regularization; [St14]) Assume that there is an integer  $n$  such that

$$n = c(D'') - \max \deg V(D'') \text{ for all } D'' \in \mathcal{D}_*. \quad (35)$$

Then (35) holds for all diagrams  $D'$  obtained by positive  $\tilde{r}_2'$ -twists and flypes from  $D$ .

The proof is a simple inductive application of the skein relation of  $V$ . We called in [St14] property (35) *regularity*, and its verification *regularization*. We will use below the same terms. (There is a similar property  $P$ , which we will encounter in some form in §7.)

We check first that  $\max \deg V(E) > 0$  for all diagrams  $E \in \mathcal{E}$ . Then split  $\mathcal{E}$  into sets  $\mathcal{E}_n$  of diagrams  $E$ , according to  $n = c(E) - \max \deg V(E)$ . (In practice the occurring values of  $n$  are between 3 and 6.) Now  $D$  is supposed to be obtained from some diagram in some  $\mathcal{E}_n$  by some number of positive  $\tilde{r}_2'$ -twists. To rule this out, build  $(\mathcal{E}_n)_*$  and check for each  $E \in (\mathcal{E}_n)_*$  that  $n = c(E) - \max \deg V(E)$ . Using the skein relation of  $V$  one can then show that this property is preserved under further positive  $\tilde{r}_2'$ -twists. So if  $D$  is a diagram not admitting a wave move, then

$$\max \deg V(D) = c(D) - n \geq c(E) - n = \max \deg V(E) > 0.$$

Thus no unknot diagram can occur as  $D$ . With this proposition 5.3 is established.  $\square$

### 5.3 Simplifications

There are a few shortcuts in the above procedure which we mention. In building  $\mathcal{E}$  from  $\mathcal{D}$ , one can also discard all semiadequate diagrams  $X$ . Semiadequacy persists under twising, and by [Th] (or by theorem 2.7) the unknot has no non-trivial semiadequate diagrams. Since we will come to talk about it in the sequel, we call this step shortly the *semiadequacy shortcut*. For knots with  $P = 1$ , it can always be applied.

When switching crossings one by one to obtain  $\hat{D}$  from  $D'$ , for  $c_-(D) = g(D) \in \{3, 4\}$ , we can use lemma 5.2. We discard special generators immediately and for non-special generators switch only crossings that have a Seifert equivalent one. For  $g = 4$ , we are left, after flyping and discarding  $\hat{D}$  with trivial clasps, with 31 possible diagrams. Using the semiadequacy shortcut, we find  $\#\mathcal{E} = 4$ , and regularization applies with  $n = 5$ .

Another shortcut comes from the Rudolph-Bennequin inequality (2). It implies the following:

**Lemma 5.5** If  $D$  is a diagram of a knot  $K$ , and  $k$  the number of  $\sim$ -equivalence classes of  $D$  which contain a negative crossing, then  $g_s(K) \geq g(D) - k$ .

**Proof.** We flype  $D$  so that between the  $n$  negative crossings in each such  $\sim$ -equivalence class, we have  $n - 1$  negative Seifert circles (of valence 2). The estimate then follows from (2).  $\square$

This means that for given  $\alpha$  (index of distribution of negative crossings into  $\sim$ -equivalence classes) of length  $k$ , one needs to consider only  $D'$  with  $g(D') \leq k$ . We call this step the *Rudolph-Bennequin shortcut*.

The reason for not including these shortcuts into the proof of proposition 5.3 is that in the modifications of the proof for the related theorems their legitimacy will vary from case to case.

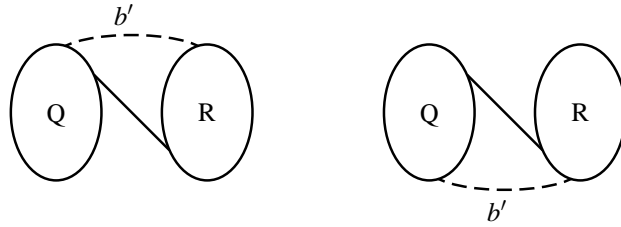
We now set out to prove theorem 1.1 in its full form. The basic point of removing flypes from proposition 5.3 lies in the construction (33) and lemma 5.3. We first improve lemma 5.3.



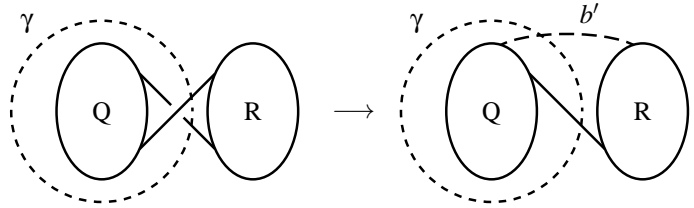
**Lemma 5.6** Assume a diagram  $D$  is given in the presentation (34), with at least 2 tangles  $R, Q$ . Assume  $D$  admits a reducing wave move shrinking a bridge  $b \subset R \cup Q \cup \{p\}$  to a bridge  $b'$ . Then this move can be chosen so that  $b$  lies entirely within  $R$  or  $Q$ , or in  $R \cup \{p\}$  or  $Q \cup \{p\}$ .

**Proof.** If  $b$  does not contain  $p$ , then  $b$  lies entirely within  $R$  or  $Q$ . So assume again that  $b$  passes  $p$ . It is obvious that  $b'$  can be chosen not to pass any crossing outside of  $R \cup Q \cup \{p\}$ . If  $b'$  passes  $p$ , then the wave move between  $b$  and  $b'$  would split into two parts, one which modifies the part of  $b$  between  $x$  and  $p$ , and another one which acts on the remaining part between  $p$  and  $y$ . At least one of these would be a reducing move, and so we are done.

Thus we may assume that  $b'$  does not pass  $p$ . Then there are 2 options how  $b'$  looks outside  $R$  and  $Q$ :



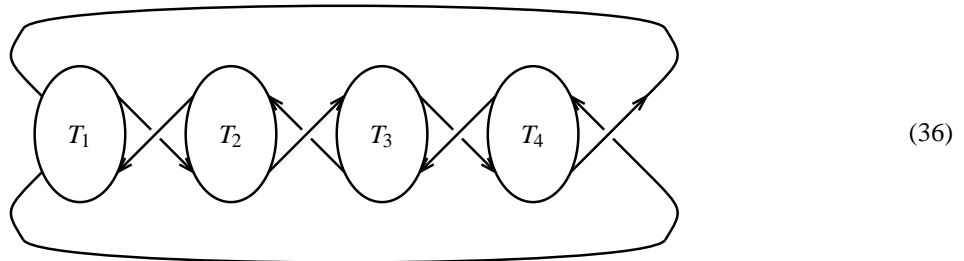
They are symmetric, so look just at the first one. By drawing a circle  $\gamma$  like



we see that again  $\gamma$  splits the wave move into two parts, for which the bridge lies within  $Q \cup \{p\}$  and  $R$  resp., and at least one of which is reducing.  $\square$

Now we need a more delicate treatment of the freedom to apply flypes at a crossing of a  $\sim$ -equivalence class. It will be convenient to slightly change the nomenclature of tangles from (34).

**Definition 5.1** Let the (flying) degree  $\deg C$  of a  $\sim$ -equivalence class  $C$  in  $D$  be the maximal number of tangles  $T_i$  in a presentation of  $D$  like (36) (where it is 4), such that no  $T_i$  contains crossings in  $C$ , and each crossing not contained in any tangle belongs to  $C$ . We call such  $T_i$  *essential* (w.r.t.  $C$ ).

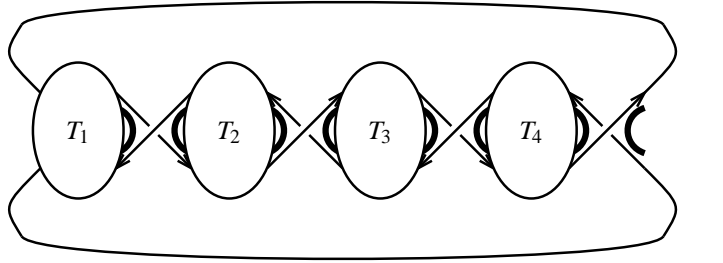


We call  $C$  (flying) *inactive* if it has degree 1. Hereby we insist on the strand orientations given in (36). Note that in a trivial  $\sim$ -equivalence class, a crossing may admit type A flypes. In that case we still regard its class as inactive.

The following lemma shows how much the degree of a  $\sim$ -equivalence class can grow. It is useful (and was used) as a consistence test during the calculation. An  $n$ -twist tangle is a tangle (with a diagram) of  $n$  crossings, containing  $n - 1$  clasps. Again we can distinguish between parallel and reverse twists if  $n > 1$ . A  $(n, m)$ -pretzel tangle is the sum of an  $n$ - and  $m$ -twist tangle, performed so that we do not obtain a  $n + m$ -twist tangle.

**Lemma 5.7** In a knot diagram of genus  $g$ , each  $\sim$ -equivalence class  $C$  has degree at most  $g$ . Moreover, if  $\deg C = g$  and  $C$  is non-trivial, then one of the  $T_i$  is a twist tangle, or a  $(n, m)$ -pretzel tangle for  $n, m$  odd (and reverse twists in either case).

**Proof.** Note that the Seifert circles in (36) (thickened below) are of the shape



We look at the Euler characteristic. First  $\chi(D) = 1 - 2g$ . We apply next the replacement

$$\begin{array}{c} \text{Seifert circle } T_i \end{array} \longrightarrow \text{single Seifert circle} \quad (37)$$

(in other words, “emptying out”  $T_i$  and making it into a single Seifert circle). One observes that, because  $T_i$  is essential, (37) augments  $\chi$  by at least 1, and by exactly 1 only if  $T_i$  is a  $(n, m)$ -pretzel tangle or a  $2n$ -twist tangle. Connectivity of  $D$  implies that in former case  $n, m$  are odd, and in either case that at most one such  $T_i$  can occur. Thus (37) for all but at most one  $i$  augments  $\chi$  by at least 2. Finally, after all instances of (37) are performed, the resulting diagram  $D_0$  is still connected, so  $\chi(D_0) \leq 1$ .

The first claim of the lemma follows then by a simple estimate. The second claim follows similarly, because if  $C$  is non-trivial, then  $\chi(D_0) = 0$ . Thus at least once in (37),  $\chi$  must go up only by 1.  $\square$

**Definition 5.2** Let  $X$  be a diagram. Fix for  $X$  a set  $C_X$ , consisting of one positive crossing in each non-negative  $\sim$ -equivalence class in  $X$ . (Non-negative means here that the class contains at least one positive crossing.) Let  $p : C_X \rightarrow \mathbb{N}$  be a function, defined on  $c \in C_X$  depending on its  $\sim$ -equivalence class  $C$  as follows:

$$p(c) = \left\{ \begin{array}{ll} 1 & \text{if } C \text{ is trivial and inactive,} \\ 0 & \text{if } C \text{ is non-trivial and inactive,} \\ 0 & \text{if } C \text{ is active and } |C| \geq \deg C, \\ \left\lfloor \frac{\deg C - |C| + 1}{2} \right\rfloor & \text{if } C \text{ is active and } |C| < \deg C. \end{array} \right\}.$$

Then let

$$X_+ = \left\{ X' : \begin{array}{l} X' \text{ is obtained from } X \text{ by } 0, \dots, p(c) \\ \text{twists at } c \text{ for each } c \in C_X \text{ and type} \\ B \text{ flypes, so that } X' \text{ has no trivial clasp} \end{array} \right\}.$$

(The choice of twists is again understood to be independent for each crossing, so that there are  $\prod_c (p(c) + 1)$  choices. Now, however,  $|X_+|$  will in general be larger, due to the flypes.) Again if  $\mathcal{X}$  is a set of diagrams, we let  $\mathcal{X}_+ = \bigcup_{X \in \mathcal{X}} X_+$ .

The next lemma now shows up to how many crossings we must twist in each  $\sim$ -equivalence class, in order to be sure that reducing wave moves will persist (without need of flypes) if more crossings are added.

**Lemma 5.8** Assume that for a generator  $D$ , every diagram in  $D_+$  admits a reducing wave move. Then so does every diagram in its series  $\langle D \rangle$  obtained by type B flypes and positive  $\tilde{t}_2$ -twists from  $D$ .

**Proof.** Let  $D' \in \langle D \rangle$ . We fix a  $\sim$ -equivalence class  $C$  in  $D'$  that contains at least one positive crossing. Then we will obtain below a diagram  $D'' \in \langle D \rangle$  from  $D'$  by flypes and removing pairs of crossings in  $C$  by undoing twists. In this diagram  $D''$  the class  $C$  will have at most  $\deg C + 1$  crossings if  $C$  is active, and at most 3 crossings if  $C$  is inactive. We will argue that if  $D''$  admits a reducing wave move, then so does  $D'$ . The lemma follows then by induction on the  $\sim$ -equivalence classes of  $D$ .

In order to find this diagram  $D''$ , first assume that  $C$  is inactive. Let  $T = T_1$  be the complementary tangle to  $C$ . We may then assume that  $C$  is positive, since otherwise we have a trivial clasp. (In the absence of flypes, one cannot put tangles “between” clasps.) If  $|C| \leq 3$ , we may set  $D'' = D'$ , so let  $|C| \geq 4$ . Now remove by undoing twists pairs of crossings in  $C$  until 2 or 3 remain. Let this be  $D''$ .

If  $D''$  admits a reducing wave move, then lemma 5.6, applied on  $T_1 = T$  and  $T_2$  consisting of the  $|C| - 2$  internal crossings in  $C$ , shows that this move avoids  $T_2$ . (Note that, unlike definition 5.1, lemma 5.6 makes no assumption whatsoever on  $T_i$ .) Then  $T_2$  can be replaced by arbitrary many crossings in  $C$ , and our claim for  $D''$  is justified.

Next let  $C$  be active. We assume that  $|C| \geq \deg C + 2$ , and that between the  $T_i$  only crossings of the same sign occur in  $D'$ . (Otherwise we have again a trivial clasp.) We call such a collection of crossings a *group*. We attach a non-zero integer to a group by saying that it is a group of  $-t_i$  if it has  $t_i$  negative crossings, and a group of  $t_i$  if it has  $t_i$  positive crossings.

If there is a triple of crossings, i.e. a group of  $t_i$  for  $|t_i| \geq 3$ , then we can undo a  $\tilde{t}_2$ -twist. Each  $T_i$  will have a crossing of the same sign on either side, so by lemma 5.6 a wave move in the simplified diagram would give a wave move in the original diagram.

If there are two pairs of crossings, then we remove one crossing in each pair, and flip the portion of the  $T_i$  on one side around. (This is a flype followed either by undoing a twist, or removing a trivial clasp.) The same argument applies, only that bridges might have turned into tunnels. (Keep in mind that we regard tunnels as equivalent to bridges.)

With this we managed to simplify  $C$  by two crossings, so to find  $D''$ , repeat this until  $|C| - \deg C \in \{0, 1\}$ .

Let us finally briefly argue why we can assume that  $D''$  has no trivial clasps. First, we assumed that in  $D'$  every group of a  $\sim$ -equivalence class is signed, and constructed  $D''$  so that it has the same property. This is why  $D''$  has no trivial reverse clasp. If  $D''$  had a trivial parallel clasp, then the two crossings in it would be in a trivial and inactive  $\sim$ -equivalence class. Thus the only difference between  $D''$  and  $D'$  in that classes could be that at the positive crossing, some  $\tilde{t}_2$ -twists are applied. We would then have a variant of the move (32) in  $D'$ .  $\square$

The following modification of the previous lemma is the analogue of the test given in the paragraph after the proof of lemma 5.3, which we apply now to avoid flypes.

**Lemma 5.9** Assume that  $D$  is a (not necessarily alternating) generator diagram, or obtained from such a diagram by one negative  $\tilde{t}_2$ -twist, and possible flypes. Assume that each diagram  $D''$  in  $D_+$  admits a reducing wave move, or  $\sigma(D'') > 0$ . Then some of these two alternatives applies to every diagram  $D' \in \langle D \rangle$  obtained by type B flypes and positive  $\tilde{t}_2$ -twists from  $D$ .

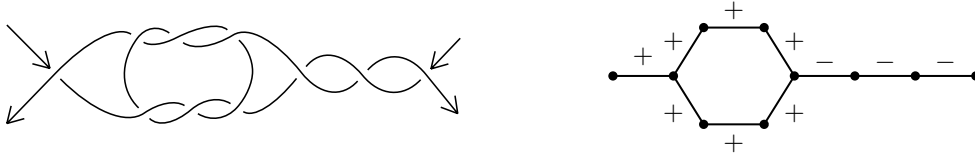
**Proof.** Remark that in the previous proof undoing a  $\tilde{t}_2$ -twist at a positive crossing, as well as a flype, does not augment  $\sigma$ .

Then we consider only the situation where we need to undo a negative  $\tilde{t}_2$ -twist. Let  $C$  again be the  $\sim$ -equivalence class modified in passing from  $D'$  to  $D''$ . Following the argument in the previous proof, we see that a negative  $\tilde{t}_2$ -twist could occur (after a possible flype) only if in  $C$  we have a group of  $t_i \leq -3$ , or two groups of  $t_i = -2$ . Then we have  $\geq 3$  negative crossings (at least one positive crossing) in  $C$ .

So assume that  $D$  has a  $\sim$ -equivalence of  $\geq 3$  negative crossings. This means, by the Rudolph-Bennequin shortcut, that we need to consider only generators of genus at most 2. The description of unknot diagrams of genus 1 is easy.

We thus assume  $g(D') = 2$ . By lemma 5.7, we have  $\deg C \leq 2$ . We assumed that  $C$  has a positive crossing, and thus if  $D'$  has no trivial clasp, at least one of the two groups in  $C$  has  $t_1 > 0$ . Because of the negative crossings in  $C$ , the other one must have  $t_2 \leq -3$ , and in particular  $\deg C = g(= 2)$ . The second part of the lemma then shows that one of the  $T_i$  is a twist tangle resp. pretzel tangle. (One can use alternatively theorem 2.12 directly.) This tangle, say  $T_1$ , is connected by a crossing in  $C$  on one side in a non-alternating way. Thus for a twist tangle  $T_1$ , we have a version of the move (32) (in which the twist tangle would have an even number of crossings and orientation would be slightly different).

For a pretzel tangle  $T_1$ , we can apply a similar move if one of  $m$  or  $n$  is equal to 1. If  $m, n \geq 3$ , we will argue that the diagram is knotted. We have in  $D'$  a tangle like



Its Seifert graph is shown on the right. We use now the inequality (18) of Murasugi-Przytycki. (The problematic cases discussed below in §7.2 do not occur here.) By successively contracting the leftmost vertex of the negative edges, we see that  $\text{ind}_-(D') \geq 3$ . Then (18), slightly rewritten using §2.2, yields

$$\min \deg P(D') \geq 2g(D') - 2c_-(D') + 2\text{ind}_-(D') \geq 2.$$

The skein polynomial thus shows that the unknot cannot occur in such a diagram.  $\square$

**Proof of theorem 1.1.** The case  $k = 1$  was known, and  $k = 2$  can be easily concluded from the proof of proposition 5.2. Let us thus assume  $k = 3, 4$ . Moreover, by theorem 2.11, we may assume  $g(D) \geq 2$ .

The work we did allows us to proceed now similarly to proposition 5.3. We consider generators of genus 2 to  $k$ . The factor slide allows us to discard composite generators, and corollary 5.1 special ones of genus  $k$ . We take the positification of one diagram per generator. We switch crossings (successively testing  $\sigma$  to save work) to find all  $k$ -almost positive diagrams of  $\sigma \leq 0$ . We must here, though, allow for crossings of opposite sign in the same  $\sim$ -equivalence class, incl. for trivial clasps.

Our understanding is that each diagram we have to treat is obtained from the grenerator diagram first by type A flypes, then  $\tilde{f}_2$ -twists and finally type B flypes. Thus first we apply type A flypes.

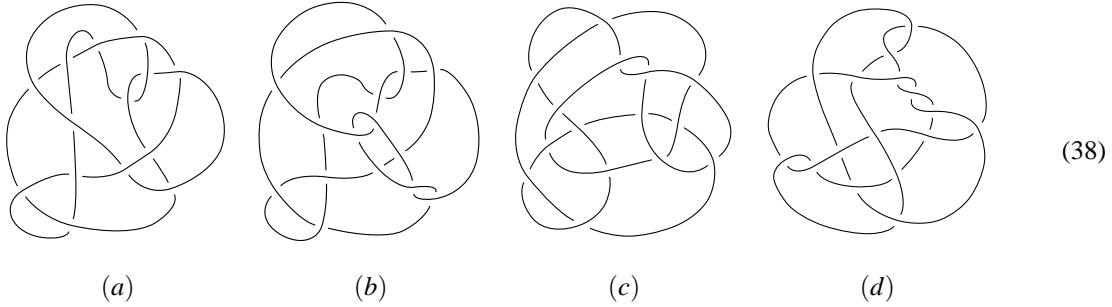
Additionally, we must consider for  $k = 4$  the diagrams, where the negative crossings split into  $\sim$ -equivalence classes (3, 1). We take all genus 2 generators, apply type A flypes, make positive, switch two crossings, and test  $\sigma \leq 2$ . Then we apply one negative  $\tilde{f}_2$ -twist, and test  $\sigma \leq 0$ .

Let  $\mathcal{D}$  be the set of diagrams that we obtained so far. Our understanding is now that every diagram we need to consider is obtained from those in  $\mathcal{D}$  by positive  $\tilde{f}_2$ -twists and type B flypes. For each diagram  $D$  in  $\mathcal{D}$ , we then replace  $D_*$  in the context of (33) by  $D_+$ . (This then removes the diagrams with trivial clasps.) We discard all diagrams in  $\mathcal{D}_+$  admitting a reducing wave move or having  $\sigma > 0$ .

Finally, we apply Jones polynomial regularization (lemma 5.4) on the remaining set  $\mathcal{E}$ . Here still  $(\mathcal{E}_n)_*$  must be used (an *not*  $(\mathcal{E}_n)_+$ ). However, now crossings of either sign may occur in a  $\sim$ -equivalence class, so the set  $C_X$  in (33) must be understood to contain one positive crossing in each  $\sim$ -equivalence class that has a positive crossing.  $\square$

## 5.4 Examples

**Example 5.1** It is to be expected that the simplification of unknot diagrams grows considerably more complicated when the number of negative crossings increases. We give a few examples illustrating this.

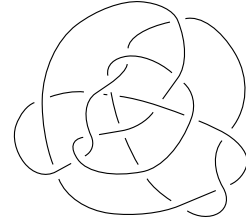


The 5-almost positive 14 crossing diagram (a) does not allow a reducing wave move, and so theorem 1.1 fails in this form for  $k = 5$ . Our example allows a reducing wave move after a flype. We checked then that up to 18 crossings, all 5-almost positive unknot diagrams still were found to admit a reducing wave move after flypes, so that at least proposition 5.3 might be true for  $k = 5$ . For 6-almost positive diagrams, even flypes are not enough. The 15 crossing diagram (b) admits a flype, but no wave move applies in any of the two diagrams obtained by flypes. The diagram (c) is similar, though now no flypes apply either. The 16 crossing diagram (d) is like (c) but is additionally special.

From the point of view of alternation, we have from theorem 1.1:

**Corollary 5.2** For  $k \leq 4$ , a special  $k$ -almost alternating unknot diagram is trivializable by crossing number reducing wave moves and factor slices.  $\square$

**Example 5.2** We made again some experiments to gain evidence as to how (in)essential is the speciality assumption is. Tsukamoto's theorem 2.6 for  $k = 1$  motivates that flypes must be included. We found that all 2-almost alternating unknot diagrams up to 18 crossings admit, after possible flypes, a reducing wave move, and in fact such a move that leads to a 2-almost alternating diagram. In contrast, there is a 3-almost alternating 17 crossing diagram (on the right) which does not admit neither a flype nor a (whatsoever) reducing wave move.



As already highlighted, both corollary 5.2 and the above computation deserve to be accompanied by the warning that wave moves do not (in general) preserve the property  $\leq k$ -almost alternating. (In particular, we did not claim in the corollary that diagrams obtained intermediately after moves during the process of reduction are either special or  $\leq 4$ -almost alternating.) Thus there seems some meaning in asking what moves in  $\leq k$ -almost alternating diagrams should naturally play the role of wave moves in  $\leq k$ -almost positive diagrams.

A common feature of these examples is that still one can simplify them by (reducing) wave moves, after first performing some crossing number preserving ones. There seems some folklore belief that this might be true in general.

**Conjecture 5.1** Crossing number preserving and crossing number reducing wave moves in combination suffice to trivialize unknot diagrams.

I had only an imprecise source for this conjecture, which also reports that it was checked up to relatively large crossing number (above 20). Within my computing capacity, I was able to confirm it up to 17 crossings. We explained already that (since factor sliding is a preserving wave move), theorem 1.1 in particular settles the conjecture for  $\leq 4$ -almost positive diagrams.

**Remark 5.1** We do not know if factor sliding is essential in theorem 1.1. The calculation we did was based on the Dowker notation [DT]. It would make a treatment of composite diagrams indefinitely more complicated, too much for the minor (potential) benefit of removing the factor sliding.

### 5.5 Non-triviality of skein and Jones polynomial

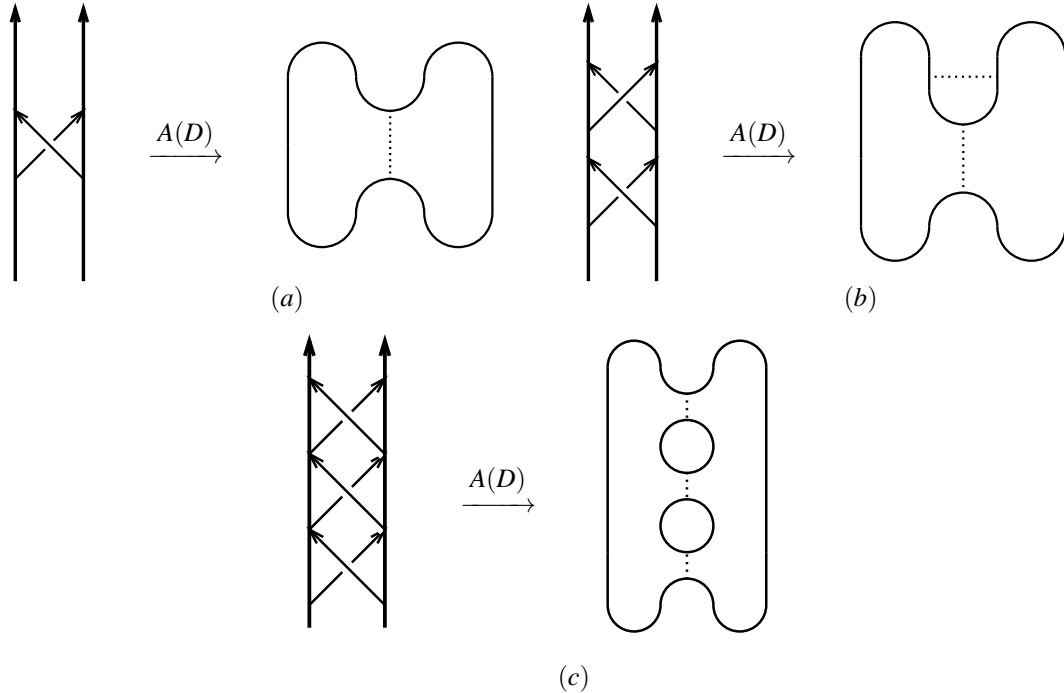
**Proof of theorem 1.2.** The non-triviality result for the skein polynomial is almost immediate from the proof of proposition 5.3. Assume  $P = 1$ . Then in particular  $V = 1$ , so  $K$  is not semiadequate by theorem 2.7, and  $\Delta = 1$ , so  $\sigma = 0$ . Also by  $\min \deg_i P = 0$  and Morton's inequalities a  $\leq 4$ -almost positive diagram  $D$  must have  $g(D) \leq 4$ . The rest of the argument and calculation is as in the proof of proposition 5.3. Here the semiadequacy shortcut is allowed, but not the Rudolph-Bennequin shortcut. This deals with the statement in theorem 1.2 about the skein polynomial.

We make now more effort for the more interesting problem regarding  $V$ . The cases  $k \leq 1$  are well-known. First assume  $k = 2$ . If  $V = 1$ , then by [St6] a 2-almost positive diagram  $D$  has  $g(D) \leq 3$ . Also  $V = 1$  implies  $\det(L) = 1$ , so that  $8 \mid \sigma(L)$ , and using  $g(L) \leq g(D) \leq 3$ , we have again  $\sigma = 0$ .

If  $D$  is composite, then one is easily out using the result for almost positive knots in [St6]. (Note that when  $\max \deg V > 0$ , then  $V$  is not a unit in  $\mathbb{Z}[t^{\pm 1}]$ ; see [J2, §12].) If  $D$  has genus 1, then  $K$  is a rational or pretzel knot, so a Montesinos knot. By [LT], it is semiadequate, and thus by theorem 2.7,  $V \neq 1$ .

So start with positive generators  $D'$  of genus 2 or 3 and  $\sigma \leq 4$ , switch two crossings to find  $\hat{D}$ , and check  $\sigma(\hat{D}) \leq 0$  and that  $\hat{D}$  is not semiadequate. We obtain again a set  $\hat{\mathcal{D}}$ . We construct again  $\mathcal{D} = (\hat{\mathcal{D}})_*$  and discard semiadequate, positive signature or wave move admitting diagrams  $X$  therein. Then we obtain again  $\mathcal{E}$  and apply a  $\max \deg V$  regularization.

For the rest of the proof we consider  $k = 3$ . By [St6] we have  $g(D) \leq 5$ . Again the composite case can be easily reduced to the prime one, so assume  $D$  is prime.



**Figure 11:** Some possible loops in the  $A$ -state of  $D$ . The left sides display the location of crossings for a pair of Seifert circles (thickened), and the right part how the loops are connected in  $A(D)$ .

**Case 1.** Clearly we must get disposed of the (computationally impossible) case  $g(D) = 5$  first.

If the 3 negative crossings are divided into Seifert-equivalence classes 1-1-1, then the situation in part (a) of figure 11 occurs 3 times; possibly the loops on the right of (a) are connected to one. (Here we must take into account that the Seifert graph of  $D$  is bipartite, so any cycle of non-parallel edges has length at least 4.) So the  $A$ -state of  $D$  has

$|A(D)| = s(D) - 3$  loops. Converting this to  $V$  using (8), we see that it contributes in degree  $m(D) = g(D) - 3 = 2$ . So  $\min \deg V \geq 2$ , and  $V \neq 1$ .

If the 3 negative crossings are divided 2-1 in Seifert-equivalence classes, with the same argument and  $|A(D)| = s(D) - 1$ , we have  $\min \deg V \geq m(D) = 1$ .

Then all 3 negative crossings are in a common Seifert-equivalence class. So we have a loop in  $A(D)$  like in part (c) of figure 11. Now if there are no more (positive) crossings in that Seifert-equivalence class,  $D$  is  $A$ -adequate. If there are such crossings, then in the  $A$ -state of  $D$  there are isolated traces connecting the same loops. By the analysis of [BMo] in lemma 2.1, we see that the coefficient vanishes in the degree in which the  $A$ -state contributes. But since  $|A(D)| = s(D) + 1$ , this degree is  $m(D) = g(D) - 5 = 0$ . So  $\min \deg V > 0$ . This finishes the case  $g(D) = 5$ .

**Case 2.** Let  $g(D) = 4$ . Again one cannot have the 3 negative crossings divided into Seifert-equivalence classes 1-1-1, because then  $\min \deg V \geq m(D) = g(D) - 3 = 1$ .

**Case 2.1.**  $D$  is special.

If all 3 negative crossings are Seifert-equivalent, then either the diagram is  $A$ -adequate (if no further crossings are Seifert-equivalent), or simplifies after a possible flype (if there are such crossings).

So assume the 3 negative crossings are divided into Seifert-equivalence classes 2-1. Now  $D$  would simplify if there is a positive Seifert-equivalent crossing to a negative one. Otherwise the class that has the single negative crossing  $p$  must be trivial. Then in the  $A$ -state of  $D$  the crossing  $p$  leaves an isolated self-trace (as in part (a) of figure 11). By lemma 2.1, we see again that the coefficient of the  $A$ -state vanishes. But since  $|A(D)| = s(D) - 1$ , this degree is  $m(D) = g(D) - 4 = 0$ . So  $\min \deg V > 0$ . This finishes the case  $g(D) = 4$ ,  $D$  special.

**Case 2.2.**  $D$  is non-special.

**Case 2.2.1.** The 3 negative crossings are Seifert-equivalent. Since we can assume  $D$  is not  $A$ -adequate, we must have a positive Seifert-equivalent crossing. Now the canonical surface of  $D$  has then a compressible Murasugi summand. By Gabai's work [Ga2, Ga3], this surface is not of minimal genus, so  $g(L) \leq 3$ . Now again, since  $8 \mid \sigma$ , the condition  $\sigma = 0$  emerges, and we can use a  $\sigma$  test.

We need to start only with positivized generator diagrams  $D'$  for which  $\sigma(D') \leq 6$ . (This drops drastically the work – see the proof of proposition 6.1.) To obtain  $\hat{D}$ , we switch 3 crossings in a Seifert-equivalence class of at least 4, and check if  $\sigma(\hat{D}) \leq 0$ . (Since  $\sigma$  is unaltered under mutation, one diagram per generator knot will do.) 10 diagrams  $\hat{D}$  remain; after discarding semiadequate diagrams 4 remain. Then we build  $\chi_*$  as in (33) for the set  $\chi$  of these 4 diagrams. After discarding diagrams in  $\chi_*$  admitting a wave move and such of  $\sigma > 0$ , the diagram set becomes empty.

**Case 2.2.2.** The 3 negative crossings are in Seifert-equivalence classes 2-1. The degree  $m(D)$  in (8) of the  $A$  state is again 0, and by lemma 2.1, we see that, for  $\min \deg V = 0$ , there must be positive crossings Seifert-equivalent to the single negative crossing in the one class (as in part (b) of figure 11), and no positive crossings Seifert-equivalent to the pair of negative crossings in the other class.

Again Gabai and  $8 \mid \sigma$  imply that  $\sigma = 0$ . So we need only the positivized generators with  $\sigma(D') \leq 6$ . It turned out that after switch of the pair of crossings in a Seifert-equivalence class of 2, we never had  $\sigma \leq 2$ . In particular we cannot build a  $\hat{D}$  by one more crossing switch so that  $\sigma(\hat{D}) \leq 0$ . So we are done.

**Case 3.**  $g(D) \leq 3$ . Now  $8 \mid \sigma$  again implies  $\sigma = 0$ . So we could proceed as for theorem 1.1, just avoiding the use of lemma 5.2.  $\square$

**Remark 5.2** For  $k = 2$  and the Alexander polynomial a similar result follows from [St7], because we showed (as likewise announced, but not given account on by Przytycki) that 2-almost positive knots of zero signature are only the twist knots (of even crossing number with a negative clasp). Later we will derive this result in a far more elegant way from our setting, using some of the work in [St12].

**Example 5.3** The  $(-3, 5, 7)$ -pretzel knot is an example of a knot with trivial Alexander polynomial, which is (by the preceding remark) 3-almost positive. We do not know whether it is the only one. (For this particular case one can, again, use the methods of this paper to easily verify 3-almost positivity directly, rather than appealing to [St7].)

**Corollary 5.3** If  $L$  is a  $\leq 4$ -almost positive smoothly slice knot, then  $V(L) \neq 1$ .

**Proof.** Let  $D$  be a  $\leq 4$ -almost positive diagram of  $K$ . Theorem 1.2 easily deals with the case that  $D$  is composite (because, as noted in its proof, if  $V \neq 1$ , then  $V$  is not a unit). Considering  $D$  to be prime, we have  $\sigma = 0$ , and lemma 5.5 implies that  $g(D) \leq 4$ . Then a calculation like in the proof of theorem 1.2 applies.  $\square$

In special diagrams, one can extend theorem 1.1 to  $k = 5$ , if one allows for flypes, and similarly the part of theorem 1.2 for the skein polynomial.

**Theorem 5.1** Let  $D$  be a  $k$ -almost special alternating knot diagram, for  $k \leq 5$ . Then the following conditions are equivalent:

- (1)  $D$  is unknotted,
- (2)  $P(D) = 1$ ,
- (3)  $D$  can be trivialized by reducing wave moves and flypes.

Remembering the diagram (d) in (38), we see that the theorem is not true for  $k = 6$ . On the other hand, for  $k = 5$  it might be true for  $k$ -almost positive diagrams (without the assumption the diagram to be special), as explained in example 5.1. The warning formulated below example 5.2 is valid here, too.

**Proof.** Clearly we need to prove only (2)  $\Rightarrow$  (3). We may further assume that  $k = 5$ , since the cases  $k \leq 4$  are contained in theorems 1.1 and 1.2. Next we may assume that  $D$  is prime, because (again, for example using (4) and properties of  $V$ ), if  $P(D)$  is a unit in  $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ , then  $P(D) = 1$ .

Since  $k = c_-(D) = 5$ , we have  $g(D) \leq 5$  by (16). Next,  $P = 1$  implies  $\Delta = 1$ , which in turn implies  $\sigma = 0$ , and the signature test is allowed. Likewise is the semiadequacy test, since  $P = 1$  implies  $V = 1$ .

Assume  $g(D) = 5$ . Since  $D$  is special, by lemma 2.2, each  $\sim$ -equivalence class is signed, or we can simplify  $D$  after a flype. By (18), since  $\text{mindeg}_l P(D) = 1$  and  $k = c_-(D) = 5$ , we must have  $\text{ind}_-(D) = 0$ . This means that there is no negative  $\sim$ -equivalence class of a single crossing. Thus the 5 negative crossings in  $D$  must lie either in a single negative class, or split into two such classes 2-3. In either case  $D$  is  $A$ -adequate, and so  $V(D) \neq 1$  by theorem 2.7.

Then we have  $g(D) \leq 4$ , and can use the generator table. If  $g(D) = 1$ , then by theorem 2.11, the claim can be checked easily. (All such  $D$  are semiadequate, except if they have a trivial clasp.)

We consider thus  $g(D) = 2, 3, 4$ . The rest of the proof is as for proposition 5.3, though with a fifth negative crossing. It may be necessary (and thus we do) consider non-special diagrams. Again, at most one negative  $\tilde{l}_2$ -twist must we dealt with. In the case of such a twist, here we must test all the generators of genus 2 and 3, and those of genus 4 whose positification has  $\sigma \leq 6$ . These cases are easily finished.

We consider the case of no negative  $\tilde{l}_2$ -twist. We proceed as before, until we build again  $\mathcal{D} = (\hat{\mathcal{D}})_*$  (and not  $(\hat{\mathcal{D}})_+$ ). Next, to obtain  $\mathcal{E}$ , we discard now not only diagrams  $D \in \mathcal{D}$  admitting a wave move, but also such that do so after a flype. (Our understanding is that we apply  $\tilde{l}_2$  moves only on non-trivial  $\sim$ -equivalence classes, so that we may use also type A flypes.)

Regularization of  $V$  applies with  $n = 6, 7, 8$ , but we obtain now for  $n = 7, 8$  a total of 330 diagrams  $E$  in  $\mathcal{E}$  on which regularization fails. We can fix this, though, by restricting the condition (35) to diagrams  $E \in \mathcal{E}$  (in our current notation) with  $\sigma(E) \leq 0$ . Then (35) is always satisfied.  $\square$

The calculation is now longer than for proposition 5.3, due to the extra negative crossing. By (18) and semiadequacy, we have restrictions also when  $g(D) \leq 4$ . We left many of these tests out, though, adopting a simplistic attitude. As soon as the calculation became manageable (say, within a couple of days), we made very little effort to speed it further up by additional tests. We were aware that every single new step put into our procedure augments the risk of error. And when all diagrams can be verified, then certainly a subset can be, too, without that we get aware of overlooking the others. Indeed, a number of iterations were necessary in order to have the single steps working well and in the correct order.



## 5.6 On the number of unknotting Reidemeister moves

There has been recently some interest in the literature to estimate the minimal number of Reidemeister moves needed for turning an unknot diagram  $D$  of  $n$  crossings into the trivial (0-crossing) diagram (see e.g. [HL, HN, Hy]). Let us call this quantity here  $r(D)$ .

It was long known from Haken theory that  $r(D)$  should be estimable from above, and in [HL] this was made explicit, by showing that

$$r(D) \leq O(C^n), \quad (39)$$

where  $n = c(D)$ , and  $O(\dots)$  means ‘at most  $\dots$  times a constant independent on  $n$ ’. Unfortunately, the exponential base  $C$  is a number with billions of digits, which renders the estimate quite impracticable.

Our above results imply the following statement, which can be regarded also as a continuation to proposition 5.2:

**Proposition 5.4** *If  $D$  is an  $n$  crossing  $k$ -almost positive unknot diagram for  $k \leq 4$  or  $k$ -almost special alternating unknot diagram for  $k \leq 5$ , then  $r(D) = O(n^p)$ , where  $p$  depends only on  $k$ .*

This is in sharp contrast to the Hass-Lagarias bound in the general case. Since in proposition 5.4 we consider only small  $k$ , we could take  $p$  also to be a global constant. Underscoring the dependence on  $k$  is made in order to indicate that our approach could work also for higher  $k$ , provided proposition 5.3 or theorem 5.1 can be extended. The values of  $p$  we will obtain for given  $k$  (the largest of which is 144) can likely be lowered by further direct calculation (as in proposition 5.2 when  $k \leq 2$ ). However, we opted for a more conceptual argument, formulating three lemmas below that have also some independent meaning.

For many diagrams  $D$  we have  $r(D) = O(n)$ . Only recently some diagrams  $D$  were found where  $r(D)$  is quadratic in  $n$  [HN]. Even although this still leaves a large gap between lower and upper bounds, we do not believe a polynomial upper bound is possible in general. We should note, though, that any set of (not-too-fancy) crossing number non-augmenting unknotting moves will yield much more tangible estimates in (39). For example, would conjecture 5.1 be true, the below lemmas 5.11 and 5.12, together with the result in [St19], would justify the constant  $C = 10.399$  in (39). See also remark 5.4.

Note also that, by lemma 5.11, the difference between studying Reidemeister moves in  $S^2$  or  $\mathbb{R}^2$  is immaterial from the point of view of proposition 5.4 (it affects  $p$  at most by  $\pm 1$ ).

Proposition 5.4 follows from the below three lemmas.

**Lemma 5.10** For fixed  $k$ , the number of  $k$ -almost positive unknot diagrams of  $n$  crossings is polynomial in  $n$ .

**Proof.** There are  $\binom{n}{k} = O(n^k)$  choices of negative crossings in an  $k$ -almost positive diagram, once the underlying alternating diagram  $\hat{D}$  is given.

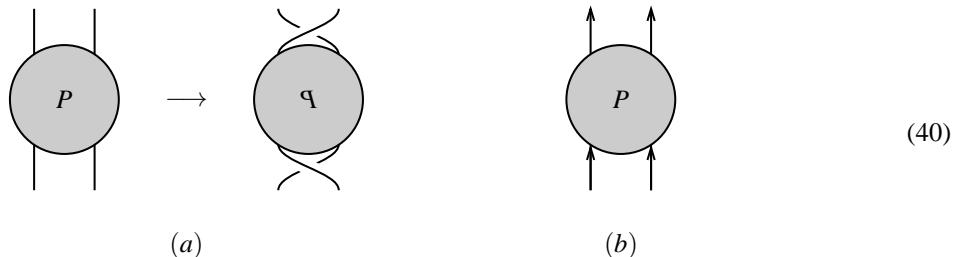
Moreover  $g = g(D) = g(\hat{D}) \leq k$ , if  $D$  is unknotted. Thus by theorem 3.1,  $D$  has at most  $6g - 3$   $\sim$ -equivalence classes. Next, by lemma 5.7, each  $\sim$ -equivalence class has at most  $g$  twist equivalence classes, so that  $\hat{D}$  has at most  $g(6g - 3)$  twist equivalence classes.

Thus the number of  $\hat{D}$  (regarded in  $S^2$ ) of  $n$  crossings is at most  $O(n^{g(6g-3)-1})$ , with  $g \leq k$ , and so the number of  $D$  is at most  $O(n^{k(6k-3)-1+k})$ .  $\square$

**Lemma 5.11** A wave move in an  $n$ -crossing diagram can be realized by  $O(n)$  Reidemeister moves.  $\square$

**Lemma 5.12** A flype in an  $n$ -crossing diagram can be realized by  $O(n^4)$  Reidemeister moves.

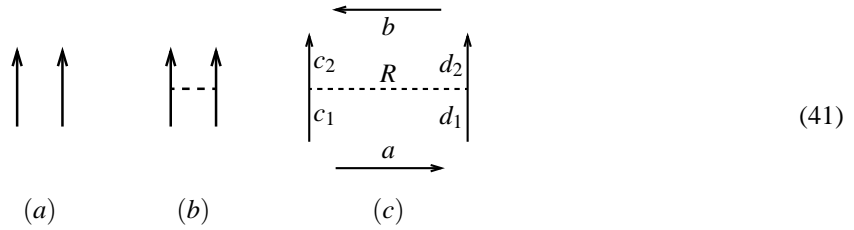
**Proof.** It is enough to prove the claim for the tangle isotopy in (a) below:



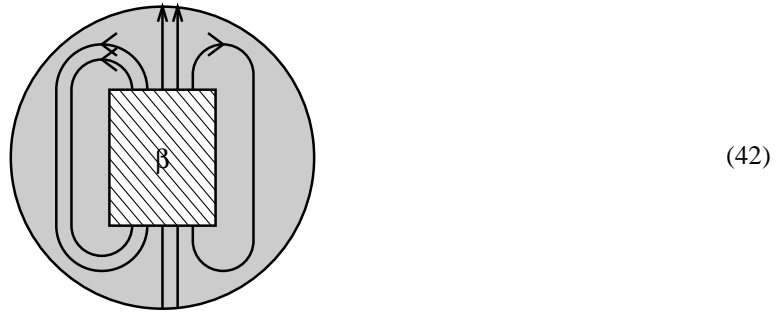
Here the second tangle is obtained by a flip (rotation by  $180^\circ$ ) on the vertical axis.

For any tangle  $P$ , there are two ways to group its 4 ends into two neighbored pairs. The results of the flips with respect to these two choices differ by two wave moves. Thus, by lemma 5.11, we may pick any of these two choices, and with this freedom we can then orient the tangle strands so that the orientation of the ends on the left of (40a) is like in (40b).

Now we would like to apply the procedure of Vogel [Vo] outside a trivial tangle (a) below,



complementary to (40b), to obtain a diagram of  $P$  as a partially closed braid  $\beta$ . (We will use below the terminology of §2.6 for braids.)



The Vogel move is a Reidemeister II move of a special type. It affects two edges in the boundary of a region  $R$ , which have the *same orientation* as seen from inside  $R$ , and belong to *distinct Seifert circles*, and creates a reverse trivial clasp inside  $R$ .

In order to realize these moves in the tangle complementary to (41a), we draw a dashed line in (41a), as in (41b), and need to be concerned only with the Vogel moves that affect this line.

Let  $R$  be the region containing the line. The two arrows in (41b) are oriented oppositely as seen from inside  $R$ , and it is easy to see that for orientation reasons, these arrows must belong to distinct Seifert circles. If now the pair  $(a, b)$  of segments in (41c) allows for a Vogel move, then at least one of the four pairs  $(a, c_1)$ ,  $(a, d_1)$ ,  $(b, c_2)$  and  $(b, d_2)$  does also. So one can find a Vogel move that avoids the dashed line.

Vogel proves that, no matter what choice of move is taken at each step, after  $O(n^2)$  applications no further moves are possible, and then the diagram is in braid form.

So we achieved the form (42) for  $P$  by  $O(n^2)$  Reidemeister moves. Since the number of Seifert circles is not changed,  $\beta$  has  $l = O(n)$  strands and  $m = O(n^2)$  crossings. (All strands of  $\beta$  are closed except 2. There is no evidence how many strands are closed on the left and on the right.)

The reverse procedure can be applied after the flip with the same cost, so it is enough to look from now on at the case that  $P$  is in the form (42).

The flip of  $P$  now transcribes into the conjugation of  $\beta$  with the half-twist  $\delta \in B_l$  (whose square is generating the center of  $B_l$ ), plus the interchange of the strands closed on the left and on the right in (42). Latter step can be accomplished by  $O(n^3)$  moves, since by lemma 5.11, moving one strand between either side takes  $O(m) = O(n^2)$  moves.

It remains to look at the flip of the braid  $\beta$ , i.e. the conjugation with  $\delta$ . With

$$\gamma_k = \sigma_1 \dots \sigma_k,$$

we can write

$$\delta = \gamma_{l-1} \cdots \gamma_1. \quad (43)$$

To simplify things slightly, let us introduce the *bands*

$$\sigma_{i,j}^{\pm 1} = \sigma_i \cdots \sigma_{j-1} \sigma_j^{\pm 1} \sigma_{j-1}^{-1} \cdots \sigma_i^{-1}, \quad (44)$$

for  $1 \leq i < j \leq l$ , so that

$$\sigma_i = \sigma_{i,i+1}. \quad (45)$$

Let

$$\pi : B_l \rightarrow S_l, \quad \pi(\sigma_i) = \tau_i = (i \ i+1)$$

be the permutation homomorphism. It now is easy to check that, *unless*  $i < k+1 < j$ ,

$$\gamma_k \sigma_{i,j} = \sigma_{\tau(i), \tau(j)} \gamma_k, \quad (46)$$

with  $\tau = \pi(\gamma_k)^{-1}$ , and that this identity is accomplished by  $O(l)$  YB relations (recall §2.6).

Now, write  $\beta$  as a word in  $\sigma_{i,j}$  using (45). Next, replace this word by  $\delta^{-1} \delta \beta$  by  $O(l^2) = O(n^2)$  Reidemeister II moves, using (43) to express  $\delta$ . Then, by using (46), one can “pull” each  $\gamma_k$  through  $\beta$  (and the situation  $i < k+1 < j$  never occurs!). Thus by  $O(m \cdot l) = O(n^3)$  applications of (46) one changes  $\delta \beta$  to  $\beta \delta$  (with  $\beta$  being  $\beta$  with  $\sigma_i$  changed to  $\sigma_{l-i}$ ), and this requires then  $O(n^3 \cdot l) = O(n^4)$  Reidemeister moves.

After that, there are two half-twists of opposite sign on the left and right closed strands, which cancel by  $O(n^2)$  moves, and the interchange of the left and right closed strands, which, as argued, costs  $O(n^3)$  moves.  $\square$

**Remark 5.3** It appears that the braid form is not essential, and in the statement of the lemma possibly  $O(n^4)$  could be made into  $O(n^3)$ . It seems, though, very awkward to write a concrete procedure down without using braids.

**Proof of proposition 5.4.** With proposition 5.3, theorem 5.1, and lemmas 5.10 to 5.12, after switching between  $O(n^{6k^2-2k-1})$  different  $k$ -almost positive unknot diagrams, each time using  $O(n^4)$ , so totally  $O(n^{6k^2-2k+3})$ , Reidemeister moves, we can reduce the crossing number. Then this procedure can be iterated. So we need at most  $O(n^{6k^2-2k+4})$  moves, that is, we can set  $p = 6k^2 - 2k + 4$  (for moves in  $S^2$ ).  $\square$

**Remark 5.4** An analogon of proposition 5.4 holds for almost alternating diagrams by theorem 2.6 and using lemmas 5.11 and 5.12. Lemma 5.10 is not true for almost alternating diagrams, but its use can be avoided. While by flypes one can in general obtain exponentially many diagrams, only a polynomial number of flypes is needed to pass between any two diagrams. This can be seen from the structure of flying circuits discussed in [ST] (see also the proof of lemma 3.1).

## 5.7 Achiral knot classification

We turn, concluding this section, to achiral knots. The proof of theorem 1.3 divides into two main cases, for prime and composite knots. For the prime case we have done the most work with the previous explanation. However, for composite knots some further arguments are necessary, the problem being that prime factors of achiral knots are not necessarily achiral. We will need some of the results in [St2] and Thistlethwaite’s work [Th].

**Proof of theorem 1.3 for prime knots.** Let  $D$  be a  $\leq 4$ -almost positive diagram and  $K$  the achiral knot it represents. We will work by induction on the crossing number of  $D$ , showing that each time we can find a wave move to simplify  $D$ , unless  $D$  has small crossing number. (For at most 16 crossings the claim of the theorem was verified in [St6].) Now assuming that  $K$  is prime, we know that if  $D$  is composite, all but one of the prime factors of  $D$  represent the unknot. Using the result for the unknot we already proved, we can trivialize these factors, and then go over to deal with the non-trivial one. With this argument we can w.l.o.g. assume that  $D$  is prime.

The proof now follows the one for theorem 1.1 with the following modifications. To determine  $\hat{D}$ , the Rudolph-Bennequin shortcut is allowed. The inequality (2) must hold for an achiral knot  $K$  by replacing  $g_s(K)$  by 0, even if

the knot  $K$  is not slice. This is because one of  $K\#K$  or  $K\#-K$  is slice, and the estimate on the right is not sensitive w.r.t. orientation and additive under (proper) connected diagram sum. The  $\sigma$  tests in determining  $\hat{D}$  are allowed because  $\sigma = 0$  by achirality.

Lemma 5.1 shows that one can apply the semiadequacy shortcut if  $c_-(D) = g(D)$  also here. So for such  $D$  we are immediately done with the verification in the proof of theorem 1.2. Even if  $c_-(D) > g(D)$ , the semiadequacy test is legitimized partly, namely if  $X \in \mathcal{D}$  satisfies  $\min \deg V(X) \geq -1$ . This can be explained as follows from the work in [Th].

If  $X$  is  $B$ -adequate, then it has the minimal number of positive crossings, which is at most 4 since  $K$  is achiral and  $\leq 4$ -almost positive. But  $X$  has also at most 4 negative crossings, so  $c(X) \leq 8$ , and we are easily out. If  $X$  is  $A$ -adequate, then positive  $\tilde{t}_2$ -twists preserve  $A$ -adequacy and  $\min \deg V$ . So  $\min \deg V(D) \geq -1$ , but  $V(D)$  is reciprocal; if  $\min \deg V \geq 0$ , then  $V = 1$  and contradiction to the semiadequacy. Then  $\min \deg V(D) = -1$ ; moreover,  $[V(D)]_{t^{-1}} = [V(D)]_{t^1} = \pm 1$  by semiadequacy. Then we use the property  $V(1) = 1$  (see [J2, §12]), which shows that only two polynomials are possible,  $-1/t + 3 - t$  and  $1/t - 1 + t$ . These are ruled out using  $V(i) = \pm 1$  and  $V(e^{\pi i/3}) \neq 0$ .

Regularization (as described at the end of the proof of theorem 1.1) applies similarly, and again we have  $3 \leq n \leq 6$ . Now  $\max \deg V \neq 0$ , but  $\max \deg V \leq 6$ , using that  $\min \deg V \geq -6$  by [St6] and reciprocity of  $V$ . So regularity shows that we can deal with all diagrams  $D$ , unless  $c(D) \leq n + \max \deg V \leq 12$ . In that case the check of [St6] for  $\leq 16$  crossings finishes the proof.  $\square$

We turn now to the two remaining knots. While  $4_1\#4_1$  is easy to deal with inductively, we will need some work for  $3_1\#13_1$ .

**Lemma 5.13** Let  $K'$  be a chiral connected sum factor of a  $\leq 4$ -almost positive achiral knot  $K$ . Then  $\max \deg_z F(K') \leq 3$ .

**Proof.** We apply (11), so in our case  $\leq 4$ -almost positivity and achirality show  $\max \deg_z F(K) \leq 7$ . Now since  $K$  is achiral but  $K'$  is not,  $K$  must have  $!K'$  as a factor too, i.e.  $K = K'\#!K'\#L$ , where  $L$  is achiral. As  $\max \deg_z F(L) \geq 0$ , we have  $7 \geq \max \deg_z F(K) \geq 2 \max \deg_z F(K')$ , and so  $\max \deg_z F(K') \leq 3$ .  $\square$

**Lemma 5.14** Let  $D'$  be a genus one digram occurring as a prime factor of  $D$ . Assume  $D'$  represents a chiral knot  $K'$ . Then this knot  $K'$  is one of the trefoils.

**Proof.** By [St4],  $K'$  is a rational or pretzel knot, so a Montesinos knot. By [LT] then we have  $\max \deg_z F(K') \geq c(K') - 2$ . By the previous lemma  $c(K') - 2 \leq \max \deg_z F(K') \leq 3$ , so  $c(K') \leq 5$ . Now  $4_1$  is achiral, and the 5-crossing knots are excluded because  $\max \deg_z F = 4$ . So  $K'$  must be one of the trefoils.  $\square$

**Proof of theorem 1.3 for composite knots.** Let  $D$  be a  $\leq 4$ -almost positive diagram and  $K$  the composite achiral knot it represents. We have  $g(D) \leq 4$ . We assume w.l.o.g. that  $D$  cannot be further simplified by flypes and wave moves.

Let first  $D$  be prime. As  $D$  admits no wave move after flypes, the proof in the prime knot case shows that  $c(D) \leq 12$ . By lemma 5.13, we have  $\max \deg_z F \leq 7$ . Then we see by check of the maximal degree of the  $F$  polynomial of achiral prime knots of at most 12 crossings that if  $K$  contains two achiral connected sum factors, then  $K = 4_1\#4_1$ . If  $K$  has a chiral factor  $K'$ , then  $K = K'\#!K'\#L$ , where  $L$  is achiral. Then we use lemma 5.13 and verify that only the trefoils have  $\max \deg_z F(K') \leq 3$  among connected sum factors of knots of  $\leq 12$  crossings, and that  $L$  must be trivial.

So for the rest of the proof we assume that  $D$  is composite. If all prime factors of  $D$  are achiral, then again we can deal with these factors separately. With the argument in the preceding paragraph,  $D$  depicts some connected sum of trefoils and figure-8-knots. Because of  $\max \deg_z F(D) \leq 7$ , there remain only the options that we want to show. So assume below that at least one prime factor of  $D$  depicts a chiral knot.

**Case 1.** Let  $D$  have first at least 3 prime factors  $D = D_1\#D_2\#D_3$ . Since  $4 \geq g(D) = \sum g(D_i)$ , we may assume w.l.o.g. that  $g(D_1) = g(D_2) = 1$ . Then the knots  $K_{1,2}$  these diagrams represent are trefoils (by lemma 5.14). If they are same sign trefoils, then  $K$  must factor as  $3_1\#3_1\#!3_1\#!3_1\#L$  for an achiral knot  $L$ , and  $\max \deg_z F \geq 8$ , a contradiction to

$\max \deg_z F(K) \leq 7$ . So  $K_{1,2}$  are trefoils of opposite sign. Since  $a_-(!3_1) = 3$  (see (9)), one of  $D_{1,2}$  has at least three negative crossings. So  $D_3$  has at most one negative crossing, but represents an achiral knot. We know, though, that non-trivial positive or almost positive knots are not achiral. So  $K = 3_1 \# !3_1$ , and we are done.

**Case 2.** Now we consider the situation that  $D = D_1 \# D_2$  has two prime factors  $D_{1,2}$ . Let  $K_{1,2}$  be the knots of  $D_{1,2}$ . We can group the mirrored chiral prime factors of  $K$  which occur on different sides of the decomposition  $D = D_1 \# D_2$  into a (possibly composite) factor knot  $K'$  of  $K$ , and assume that  $K_1 = K' \# L$  and  $K_2 = !K' \# M$  with  $L$  and  $M$  achiral (and possibly trivial) and  $K'$  chiral (and non-trivial by the assumption preceding the case distinction). We have by lemma 5.13 that  $\max \deg_z F(K') \leq 3$ .

**Case 2.1.**  $D_1$  is positive,  $D_2$  is  $\leq 4$ -almost positive.

**Case 2.1.1.**  $g(D_1) = 1$ . Then  $g(K' \# L) = 1$  so  $L$  is trivial, and  $K'$  is a positive knot of genus 1. By lemma 5.14 it is the positive (right-hand) trefoil. Now consider  $D_2$ , with  $g(D_2) \leq 3$ .

**Case 2.1.1.1.**  $g(D_2) = 1$ . Then by lemma 5.14,  $K_2$  is a trefoil, and by achirality of  $K$  it must be negative, so we are done.

**Case 2.1.1.2.**  $g(D_2) = 3$ . Since  $D_2$  is  $\leq 4$ -almost positive, the left Morton inequality in (16) shows  $\min \deg_l P(D_2) \geq -2$ . Moreover  $D_1$  is positive of genus 1, so  $\min \deg_l P(D_1) = 2$ . It follows that  $\min \deg_l P(D) \geq 0$ . Now  $K$  is achiral, and then [LM, proposition 21] shows  $P(D) = 1$ . But the trefoil polynomial must divide  $P(D)$ , a contradiction.

**Case 2.1.1.3.**  $g(D_2) = 2$ . We repeat the argument in the previous case. Now  $\min \deg_l P(D_2) \geq -4$ , so  $\min \deg_l P(D) \geq -2$ . The case  $\min \deg_l P(D) \geq 0$  is ruled out as before, so assume that  $\min \deg_l P(D) = -2$ , and then  $\max \deg_l P(D) = 2$ . Again  $K = K' \# !K' \# M$ , with  $M$  achiral. Comparison of the  $P$ -degrees and [LM, proposition 21] shows then that  $P_M = 1$ . So we seek an achiral knot  $M$  with trivial skein polynomial, whose connected sum with a negative trefoil  $!K'$  is a knot  $K_2$  in a  $\leq 4$ -almost positive genus 2 diagram  $D_2$ . However, in that case  $V(M) = 1$ , so  $\max \deg V(K_2) = -1$ , and also  $\sigma(K_2) < 0$ . Then, assuming w.l.o.g. that we reduced  $D_2$  by flypes and wave moves, we checked in the proof of theorem 1.1 that  $c(D_2) - \max \deg V(D_2) = n \leq 6$ , so  $c(D_2) \leq 5$ . Then clearly  $M$  is trivial, and we are done.

**Case 2.1.2.**  $D_1$  is positive and  $g(D_1) = 2$ . In the proof, given in [St2], of the fact that positive knots of genus 2 have minimal positive diagrams, we verified that all positive diagrams  $D_1$  of genus 2 can be reduced to diagrams  $D'_1$  with  $\max \deg Q(D'_1) \geq c(D'_1) - 2$ . Now  $\max \deg Q(D'_1) \leq \max \deg_z F(D'_1) \leq \max \deg_z F(D) \leq 7$ . So  $c(K_1) \leq c(D'_1) \leq 9$ , and  $K'$  is a chiral knot with  $\max \deg_z F(K') \leq 3$  occurring as a factor of a knot  $K_1$  with at most 9 crossings. Then again  $K'$  is a trefoil. But then, looking at the signature  $\sigma$ , and using that  $L$  is achiral, we have  $\sigma(D_1) = \sigma(K') + \sigma(L) = \pm 2$ , while for a positive diagram  $D_1$  of genus 2 we showed in [St2] that  $\sigma = 4$ . This contradiction finishes the case.

**Case 2.1.3.**  $D_1$  is positive and  $g(D_1) = 3$ . So  $g(D_2) = 1$  and by lemma 5.14,  $K'$  is a trefoil (and  $M$  is trivial). Now again  $\sigma(D_1) \geq 4$  by [St3], while  $\sigma(D_2) = \sigma(K') = \pm 2$ , and  $\sigma(D) = \sigma(D_1) + \sigma(D_2) > 0$ , a contradiction.

**Case 2.2.**  $D_1$  is almost positive,  $D_2$  is  $\leq 3$ -almost positive.

**Case 2.2.1.**  $g(D_2) = 3$ . Then  $g(D_1) = 1$  and  $K'$  is a trefoil, which must be right-hand, since  $D_1$  is almost positive. Using (16), we have  $\min \deg_l P(D_2) \geq 0$ , so  $\min \deg_l P(D) > 0$ , and a contradiction to achirality.

**Case 2.2.2.**  $g(D_2) = 2$ . By Morton's inequality  $\min \deg_l P(D_2) \geq -2$ . Also we know from [St2] that there is no almost positive knot of genus 1, so if  $g(D_1) = 1$ , then  $K_1$  is positive. Then by Morton  $\min \deg_l P(D_1) \geq 2$ , and  $\min \deg_l P(D) \geq 0$ . So  $P(D) = 1$ . However, if  $\min \deg_l P(D_1) \geq 2$ , then  $P(D_1)$ , which divides  $P(D)$ , cannot be a unit in  $\mathbb{Z}[t^{\pm 1}, m^{\pm 1}]$  by [LM, proposition 21], and we have a contradiction.

**Case 2.2.3.**  $g(D_2) = 1$ . Then by lemma 5.14,  $!K'$  is a trefoil (and  $M$  is trivial). It must be a negative trefoil, because otherwise  $\sigma(D_2)$  and  $\sigma(D_1)$  are both positive, and  $\sigma(D) = \sigma(D_1) + \sigma(D_2) > 0$ . So  $K'$  is a positive trefoil.

**Case 2.2.3.1.** If  $g(D_1) = 1$ , we are easily done by lemma 5.14.

**Case 2.2.3.2.** If  $g(D_1) = 3$ , then by Morton's inequality  $\min \deg_l P(D) \geq 0$ , and we have a contradiction along the above lines in case 2.1.1.2.

**Case 2.2.3.3.** So assume  $g(D_1) = 2$ . Now  $D_1$  contains a positive trefoil factor. By Morton,  $\min \deg_l P(D) \geq -2$ , so by achirality and [LM, proposition 21], and then along the lines in case 2.1.1.3, we see that  $L$  must have trivial skein polynomial. So  $K_1$  has the trefoil polynomial, and an almost positive diagram  $D_1$ . Then by [St5], we have

$2 = \max \deg_m P(K_1) = 2g(K_1) = 2g(K') + 2g(L)$ . (One can use alternatively  $\Delta$  and the theorem that  $\max \deg \Delta = 1 - \chi$  in [St14].) This implies that  $L$  must be trivial, and we are done.

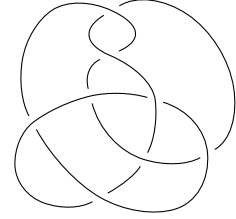
**Case 2.3.**  $D_1$  and  $D_2$  are both 2-almost positive.

If  $g(D_1) = 1$ , then ( $L$  is trivial and)  $K'$  is a chiral knot with  $\max \deg_z F \leq 3$  and a 2-almost positive genus 1 diagram. Up to reducible diagrams or diagrams with trivial clasps (which we can exclude) there are no such knots. Similarly we argue if  $g(D_2) = 1$ . So assume  $g(D_1) = g(D_2) = 2$ . Then by Morton's inequality  $\min \deg_l P(D_{1,2}) \geq 0$ , and the meanwhile well-known way how to obtain a contradiction.

The case distinction and the proof are complete. □

The proof of theorem 1.3 suggests that, beyond the description of the occurring knots, we can also obtain a partial simplification statement for their diagrams. We make this explicit in the following way.

**Corollary 5.4** Let  $K$  be a  $k$ -almost positive achiral knot for  $k \leq 4$ , and  $K \neq 3_1 \# 13_1$ . Then one can simplify a  $\leq 4$ -almost positive diagram  $D$  of  $K$  by reducing wave moves and factor slides either to a minimal crossing diagram, or to the 8 crossing diagram of  $K = 6_3$  shown on the right.



**Proof.** The exclusion of  $K = 3_1 \# 13_1$  was made in order to spare us the technical complications of dealing with non-amphicheiral prime factors. Thus we can again consider w.l.o.g. prime diagrams.

Now the proof of theorem 1.1 can be carried through, with the only difference coming in the Jones polynomial regularization of  $\mathcal{E}_n$  at the very end. (As before, the cases  $c_-(D) \leq 2$  and  $g(D) = 1$  can be checked directly.)

We can discard all alternating diagrams  $E$  in  $\mathcal{E}$ , since  $\tilde{r}_2$ -twists preserve alternation, and these are then minimal crossing diagrams, as we claimed. From the rest we check that

$$\max \deg V(E) + \min \deg V(E) \geq -1. \quad (47)$$

(It is enough to consider only  $E \in \mathcal{E}$  with  $\max \deg V(E) \leq 6$ .)

The property (47), together with the growth of  $\max \deg V(E)$  under twists (which was the subject of regularization), implies that  $V$  will not be self-conjugate on diagrams that are obtained by positive  $\tilde{r}_2$ -twists from diagrams  $E \in \mathcal{E}$ .

It is thus enough to look at  $E$  themselves. We identify all diagrams that depict achiral knots, and this leads to the shown 8 crossing diagram of  $6_3$  (which admits no reducing wave move, and a few diagrams of  $3_1 \# 13_1$ , which we chose not to dwell upon). □

From the work in §5.6 we have then also:

**Corollary 5.5** Let  $K$  be a  $k$ -almost positive achiral knot for  $k \leq 4$ , and  $K \neq 3_1 \# 13_1$ . Then one can simplify an  $n$ -crossing  $\leq 4$ -almost positive diagram  $D$  of  $K$  to a minimal crossing diagram by  $O(n^p)$  Reidemeister moves, where  $p$  depends only on  $k$ . □

Apart from the generator compilation, the most decisive reason for the success of such proofs is the easy comutability of the invariants we use, primarily the signature  $\sigma$  and Jones polynomial  $V$ . It still took some days of work on a computer to complete the proofs. However, rather than thinking of such an approach as tedious, we feel that it should be considered paying high tribute to this efficiency, which makes it not too hard to evaluate the invariants on thousands of diagrams. (The most complicated ones we encountered have 35 crossings.) Recently, a lot of attention is given to “relatives” of  $\sigma$  and  $V$  defined in terms of homology theory. The comutability of these new invariants is still sufficiently difficult, so that performing with them a procedure similar to ours seems, unfortunately, out of question for quite a while.

## 6 The signature

As a last application related to positivity, we can finally settle a problem, initiated in [St7], and whose solution has been suspected for a while. First we have

**Proposition 6.1** The knot  $14_{45657}$  (depicted in figure 4 of [St7]) is the only positive knot of genus 4 with  $\sigma = 4$ . It has only one positive (reduced) diagram, its unique 14 crossing diagram.

This is the first *non-alternating* positive knot, which now is known to have only one positive diagram.

**Proof.** Let us seek a positive diagram of such a knot. Series of composite and special generators have  $\sigma \geq 6$  and are clearly ruled out. Checking the  $\sigma$  of the positively crossing-switched non-special generators reveals that  $\sigma = 8$  occurs 1,927,918 times,  $\sigma = 6$  occurs 6662 times, and  $\sigma = 4$  only once, for the positive diagram of  $14_{45657}$ . Applying a  $\tilde{t}_2$  twist at whatever crossing of this diagram gives  $\sigma = 6$ .  $\square$

In general, the problem what is the minimal signature of positive knots of some genus is very difficult to study. We devoted to it a separate paper [St12]. Some of the theoretical thoughts there can be applied in practice here to prove

**Theorem 6.1** The positive knots of  $\sigma = 4$  are:

- 1) all genus 2 knots,
- 2) an infinite family of genus 3 knots, which is scarce, though, in the sense that asymptotically for  $n \rightarrow \infty$  we have

$$\frac{\#\{K : K \text{ positive}, g(K) = 3, \sigma(K) = 4, c(K) = n\}}{\#\{K : K \text{ positive}, g(K) = 3, c(K) = n\}} = O\left(\frac{1}{n^{10}}\right), \quad (48)$$

and

- 3) the knot  $14_{45657}$  (of genus 4).

**Proof.** The genus 2 case is obvious, and let us first argue briefly about the estimate (48) for genus 3. First, by the result of [SV] we know that the number of special alternating knots of genus 3 and  $n$  crossings behaves asymptotically like a constant times  $n^{14}$ . That these are asymptotically dense in the set of positive genus 3 knots follows from the result of [SV] that maximal generators are special alternating, and an estimate in [St8], which shows that

$$c(D) - c(K) \leq 2g(K) - 1, \quad (49)$$

for a positive diagram  $D$  of a (positive) knot  $K$ . This argument determines the behaviour of the denominator on the left of (48) for  $n \rightarrow \infty$ .

To estimate (asymptotically, from above) the numerator, we use again (49). This basically (up to a constant) allows one to go over from counting positive knots to counting positive diagrams. Now the positive diagrams of genus 3 with  $\sigma = 4$  were described in §4 of [St7] (see the remarks following proposition 4.1 there). We know that we have infinite degree of freedom to apply  $\tilde{t}_2$ -twists in at most 5  $\sim$ -equivalence classes. Thus the number of relevant diagrams of  $n$  crossings is  $O(n^4)$ , and the behaviour of the numerator of (48) is also clarified.

With this argument, and proposition 6.1, for the rest of the proof we can assume we settled  $g \leq 4$ , so we want to show that there is no positive knot of genus  $g \geq 5$  with  $\sigma = 4$ .

Let  $D$  be a positive knot diagram. Then we consider a sequence of diagrams  $D = D_0, \dots, D_n = \bigcirc$ , which is created as follows:

- 1) (“generalized clasp resolution”) If  $D_i$  has two equivalent crossings (i.e. two crossings that form a clasp, parallel or reverse, after flypes; see definition 2.5) then we change one of these two crossings, apply a possible flype that turns the crossings into a trivial clasp, and resolve the clasp. (This is the move (17) of [St12].) If there is no flype necessary, nugatory crossings may occur after the clasp resolution. Then possibly remove these nugatory crossings to obtain  $D_{i+1}$ .
- 2) (“shrinking a bridge”) If  $D_i$  has no two equivalent crossings, then we choose a piece  $\gamma$  of the line of  $D_i$  (with no self-intersections), such that the endpoints of  $\gamma$  lie in neighbored regions of  $D_i \setminus \gamma$ . This means that after proper crossing changes on  $\gamma$ , so that it becomes a bridge/tunnel, we can apply a wave move that shrinks it to a bridge/tunnel  $\gamma_1$  to length 1 in  $D_{i+1}$ . We choose between bridge/tunnel so that the resulting single crossing of  $\gamma_1$  is positive. Herein we assume that the length of  $\gamma$ , which is the number of crossings it passes, is bigger than 1. We call such a curve *admissible*. Among all such possible admissible  $\gamma$  (we showed in [St12] that some always exist) we choose one of minimal length. If several minimal length bridge/tunnels  $\gamma$  are available, we choose  $\gamma$  so that  $g(D_i) - g(D_{i+1})$  is minimal.

Now assume there is a  $D$  with  $\sigma(D) \leq 4$  and  $g(D) \geq 5$ . We will derive a contradiction. By considering the proper diagrams of the sequence found for  $D$ , we may assume w.l.o.g. that  $D_1 = D'$  has genus  $\leq 4$  and  $\sigma(D') \leq 4$ . (Note that always  $\sigma(D_{i+1}) \leq \sigma(D_i)$ .)

If  $D'$  differs from  $D$  by a clasp resolution, then  $g(D) \leq g(D') + 1$ . So the only option is that  $g(D) = 5$  and  $g(D') = 4$ . Since  $\sigma(D') \leq 4$ , we checked that  $D'$  is the diagram of  $14_{45657}$ . (We write below the knot for its positive diagram, since latter is unique.) Then  $D$  is obtained from  $D'$  by creating possibly first a certain number of nugatory crossings, and then a clasp (so that nugatory crossings become non-nugatory), and an optional flype (if there were no nugatory crossings). It is easy to see that the previously nugatory crossings can be switched in  $D$  so that the clasp to become one that does not require nugatory crossings to be added in  $D'$ . Thus if we check that all positive diagrams obtained from  $14_{45657}$  by adding a clasp *without* nugatory crossings have  $\sigma = 6$ , we are done. Now, if the genus remains 4, we already checked it. If  $g = 5$ , we have a prime positive diagram of 16 crossings. Thus we seek a prime ([O]) positive knot of  $g = 5$  and  $\leq 16$  crossings. If the knot is alternating, then by [Mu5] we have  $\sigma = 10$ , so consider only non-alternating knots. A pre-selection from the table of [HT] using the (necessary) skein polynomial condition  $\min \deg_l P = \max \deg_m P = 10$ , shows that all these knots have  $\sigma \geq 6$ .

For the rest of the argument assume that  $D'$  is obtained from  $D$  by shrinking a bridge. We would like to show that this second move is needed only in very exceptional cases, and they do not occur here.

We proved in [St12] that  $D'$  has at most one  $\sim$ -equivalence class of  $\geq 3$  elements, and they are 3. Now we use the argument in that proof to show even stronger restrictions.

**Lemma 6.1** (1) Assume  $D'$  has  $\geq 2$  disjoint clasps (meaning that the pairs of crossings involved are disjoint; the clasps may be parallel or reverse, but *not* up to flypes). Then  $g(D) \leq g(D') + 1$ .

(2) Assume  $D'$  has  $\geq 3$  disjoint clasps. Then  $D'$  cannot be obtained from a diagram  $D$  by shrinking a minimal bridge.

**Proof.** (1) There is at least one clasp  $a$  that does not contain the crossing of the shrunk bridge/tunnel  $\gamma_1$ . Since  $D$  has no equivalent crossings by assumption, the curve  $\gamma$  must pass through the clasp  $a$ . We argued in [St12] that it intersects the two edges of the clasp only once. Then  $D$  has an admissible curve of length 3, and shrinking this curve to a one crossing curve reduces the genus by at most 1. By choice of admissible curve, we have the first claim.

(2) There are at least two clasps that do not contain the crossing of the shrunk bridge/tunnel  $\gamma_1$ . Since  $D$  has no equivalent crossings by assumption, the curve  $\gamma$  must pass through both clasps. Then it intersects at least 4 edges, but we saw that at each clasp we have an admissible curve of length 3 in  $D$ . This is a contradiction to the minimality of the admissible curve.  $\square$

The rest is a simple electronic check. We determined the prime generators of genus 2 (they are 24) and 3 whose positification has  $\sigma = 4$  (they are 13, and described in [St7]). Apply flypes to them, an optional  $\tilde{t}_2^2$  twist, and again flypes. Then check that all the resulting diagrams have at least 2 disjoint clasps, and we see that  $g(D) \leq 4$  by part (1) of the lemma, so are done. The composite generators are only of genus 2 and directly ruled out the same way. Likewise, the diagram of  $14_{45657}$  has 4 clasps, and we apply part (2) of the lemma. This completes the proof of theorem 6.1.  $\square$

**Remark 6.1** We remarked in [St15] the relation of the sequence of moves we described to Taniyama's partial order. Indeed, one can apply the above sort of proof in that context. For example, one can show that  $4_1$  dominates all knots except connected sums of  $(2, n)$ -torus knots.

**Proposition 6.2** If  $K$  is an almost positive knot of genus  $g \geq 3$ , then  $\sigma \geq 4$ .

**Proof.** If the almost postive diagram is composite, we can conclude the claim from the fact that  $\sigma > 0$  when a knot is almost positive. So assume  $D$  is prime. Since  $g(K) \geq 3$ , also  $g(D) \geq 3$ . We make almost postive the 13 generators with  $\sigma = 4$  and  $14_{45657}$ . In latter case always  $\sigma = 4$ , and in former case this is also true, up to 8 diagrams of  $10_{145}$ , which has  $g = \sigma = 2$ .



Now we build for these 8 almost positive diagrams  $E$  the set  $E_*$  as in (33). Again if the negative crossing is equivalent to a positive one, we can discard the case by assuming we work with a least crossing almost positive diagram, or because the diagram becomes positive. We can also work with one diagram per generator, because the signature is mutation invariant, and so is the genus of the knots if their mutated diagrams are almost positive, as can be concluded easily from the work in [St5]. Now we check that for all diagrams in all  $E_*$  we have  $\sigma \geq 4$  or that  $\max \deg_m P = 2$ . Latter option shows by [St5] that  $g = 2$ , and the work there shows also that  $g$  will not change under  $\bar{t}_2'$  twists at a (positive)  $\sim$ -equivalence class of more than 1 crossing (in an almost positive diagram). Since all diagrams we need to check are obtained from some diagram in  $E_*$  by twists at a  $\sim$ -equivalence class of more than 2 crossings, we see that if  $\sigma = 2$ , then  $g = 2$ .  $\square$

**Corollary 6.1** If  $K$  is a 2-almost positive knot, then  $\sigma > 0$  except if  $K$  is a (non-positive) twist knot.

**Proof.** If  $D$  is a 2-almost positive prime diagram of genus at least 3, then we can switch one more crossing in the above set of 8 diagrams of  $10_{145}$  and check that  $\sigma = 2$ . It remains to deal with  $g(D) \leq 2$ , and this was done in §6 of [St2] (see remark 6.1 therein). The composite case  $D$  follows again easily from the prime one.  $\square$

Even though the check in §6 of [St2] is somewhat tedious, we must consider this proof as a considerable simplification of, and far more elegant than the one in [St7]. It was the lack of such proof that was so bothering while writing [St7].

## 7 Braid index of alternating knots

### 7.1 Motivation and history

We tried to apply a variant of the regularization of lemma 5.4 to the skein polynomial. Our motivation was to address the problem studied by Murasugi [Mu4] on the sharpness of the MWF inequality (15) on alternating links.

If we have equality in (15), then one can determine the braid index from the skein polynomial. Observably this often occurs. However, the examples of unsharpness, albeit sporadic, are diversely distributed, and it seems very difficult to make meaningful statements as to nice classes on which the inequality would be sharp. It was a reasonable conjecture that it would be so on alternating links. Murasugi worked on this conjecture, and proved that it holds if the link is rational or fibered alternating. His results remain the most noteworthy ones. Then he and Przytycki proved in [MP] the cases when  $|\max \text{cf} \Delta| < 4$ , but found a counterexample of a 15 crossing 4-component link and an 18 crossing knot.

The extreme paucity of such examples (see proposition 7.1) suggests still that sharpness statements are possible for large classes of alternating links. In that spirit, we show theorem 1.4. Note that the Murasugi-Przytycki link counterexample is of genus 3, so that the theorem does not hold for links in this form. Their knot counterexample has genus 6.

To prove theorem 1.4, we need first to have a good control on the MWF bound. We would like to have a growth in degree under  $\bar{t}_2'$ -twists, and we can formulate analogous initial conditions to the case of  $V$  that ensure inductively under the skein relation such degree growth. Originally this enabled us to deal with genus 2 and, with considerable effort, genus 3. However, we always needed the initial two twist vectors for each  $\sim$ -equivalence class. The resulting tremendous growth of the number of generators and crossings in the initial diagrams made the case of genus 4 intractable, except in special cases.

Then Ohya's paper [Oh] turned our attention to [MP]. There an efficient graph theoretic machinery is developed to study the MWF bound. With the help of this machinery we were able to considerably improve our work and deal with genus 4 completely, and this is what we describe in §7. (We will, however, unlikely be able to do so for genus 5.) Later, in §8, we will show how to modify the arguments to obtain also a Bennequin surface.

The following clarification is to be put in advance. During our study of [MP], we found a gap, which is explained in §7.2. It occurred when we wanted to understand the diagram move of Figure 8.2 of [MP]. Murasugi-Przytycki seem to assume that Figure 8.2 is the general case, but we will explain that it is not. And taking care of the missing case leads to a modified definition of index, which we call  $\text{ind}_0$  (see definition 7.3). Roughly speaking, the correction

needed is that in certain cases some edges in  $\text{star } v$  are not contracted (cf. definition 2.3). So Murasugi-Przytycki's diagram move just proves instead of (19) that

$$b(L) \leq s(D) - \text{ind}_0(D). \quad (50)$$

Then the question is how do  $\text{ind}(D)$  and  $\text{ind}_0(D)$  relate to each other. We will argue that

$$\text{ind}(D) \leq \text{ind}_0(D), \quad (51)$$

which justifies (19) (and its applications in Murasugi-Przytycki's Memoir). After we found this argument, we speculated, based on our computational evidence, whether in fact always

$$\text{ind}(D) = \text{ind}_0(D). \quad (52)$$

Later this was indeed established by Traczyk, who proved in [Tr2] the reverse inequality to (51). Still our (much more awkward) definition of index needs (at least temporary) treatment, in order to prove (51) or (52) and fix the gap in [MP]. (A minor modification of  $\text{ind}_0$  will also be used for the Bennequin surfaces.) Also, one must realize that Murasugi-Przytycki's definition of index loses its geometric meaning *per sé*. It simplifies the true transformation of the Seifert graph under their diagram move, in a way which is *a priori* incorrect but (fortunately) *a posteriori* turns out to still give the right quantity. If one likes to keep the correspondence between (Seifert) graph and diagram, one must live with the circumstance that (in general) not all of  $\text{star } v$  is to be contracted.

## 7.2 Hidden Seifert circle problem

Recall definition 2.3. Now we must understand the move of Murasugi-Przytycki that corresponds to the choice of a simple edge  $e$  and the contraction of the star of  $v$  in  $G$ . (To set the record straight, we should say that this move was considered, apparently simultaneously and independently, also by Chalcraft [Ch], although merited there only with secondary attention. With this understanding, we will refer to it below still as the Murasugi-Przytycki move.) This move is shown in figure 8.2 of [MP]. Let  $D$  be the diagram before the move and  $D'$  the diagram resulting from it. Let us for simplicity identify an edge with its crossing and a vertex with its Seifert circle (see the remark above proposition 2.1). In this language, the move of Murasugi-Przytycki eliminates one crossing, corresponding to  $e$ . The crossings of the other edges  $e' \neq e$ , incident to  $v$ , do not disappear under the Murasugi-Przytycki move. Instead, they become in  $D'$  parts of join factors of  $\Gamma(D')$  that correspond to a Murasugi summand on the opposite side of the modified Seifert circle. See the proof of lemma 8.6 in [MP] and figures 12 and 13.

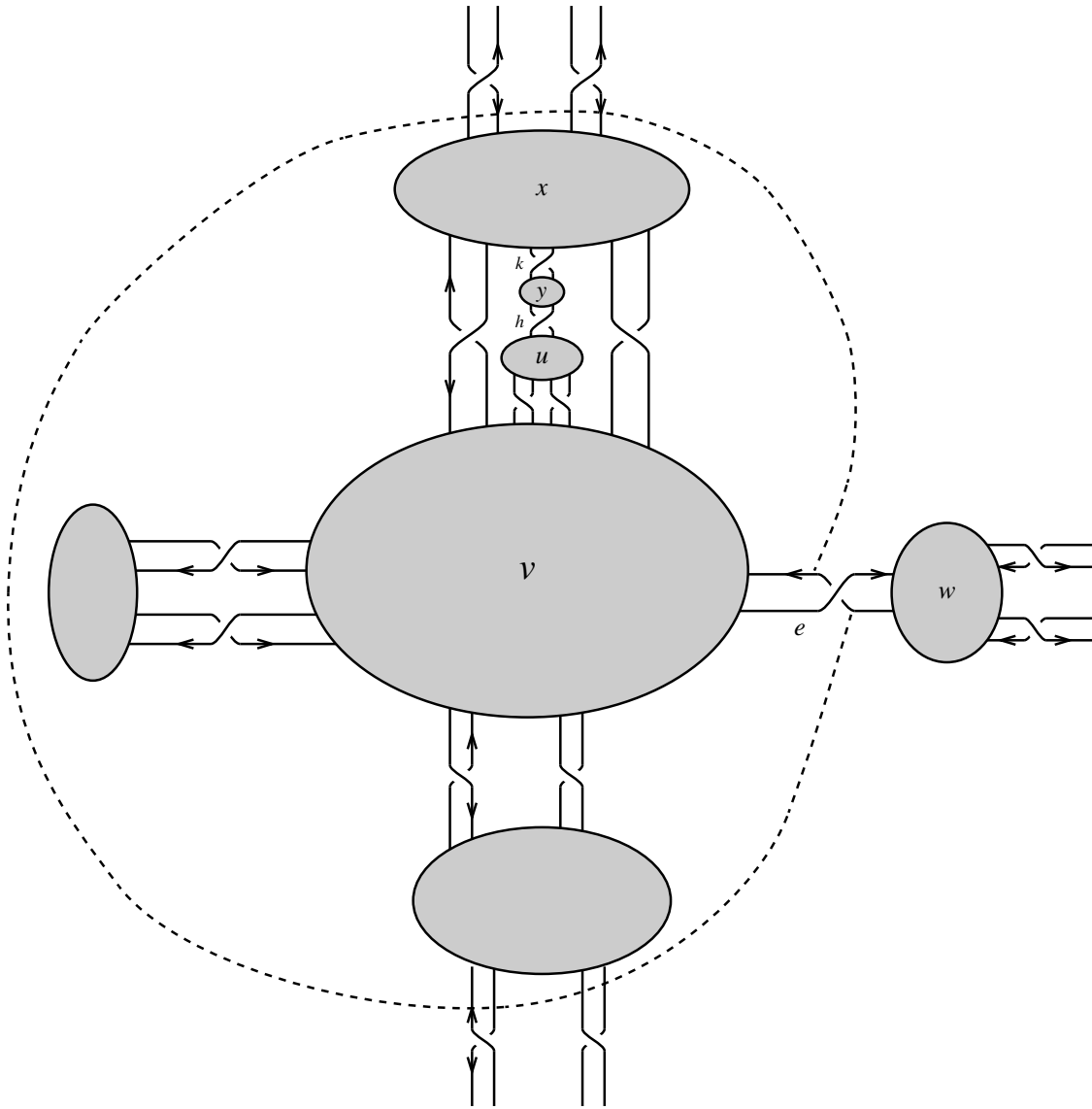
The subtlety, which seems to have been overlooked in the proof of [MP], is illustrated in figures 12 and 13. The Seifert circles adjacent to  $v$  may be nested in  $D$  in such a way that relaying the arc of  $v$  by the move, one does (and can) *not* go along *all* Seifert circles adjacent to  $v$ . In the Seifert graph  $G' = \Gamma(D')$  of  $D'$  some of the edges incident to  $v$  in  $G = \Gamma(D)$  may not enter, as written in the proof of lemma 8.6 in [MP], into block components that are 2-vertex graphs (with a multiple edge).

Still we see that contracting the star of  $v$  in  $G = \Gamma(D)$ , we obtain a graph  $\tilde{G} = G/v$ , which is a contraction of  $G' = \Gamma(D')$ . (We will later describe *exactly* how  $G'$  is constructed from  $G$ , but let us for the time being use the easier to obtain  $G/v$  instead.) Here contraction of a graph means that  $\tilde{G}$  is obtained from  $G'$  by contracting some (possibly several or no) edges, and we allow multiple edges in  $G'$  to be contracted (by doing so simultaneously with all simple edges they consist of).

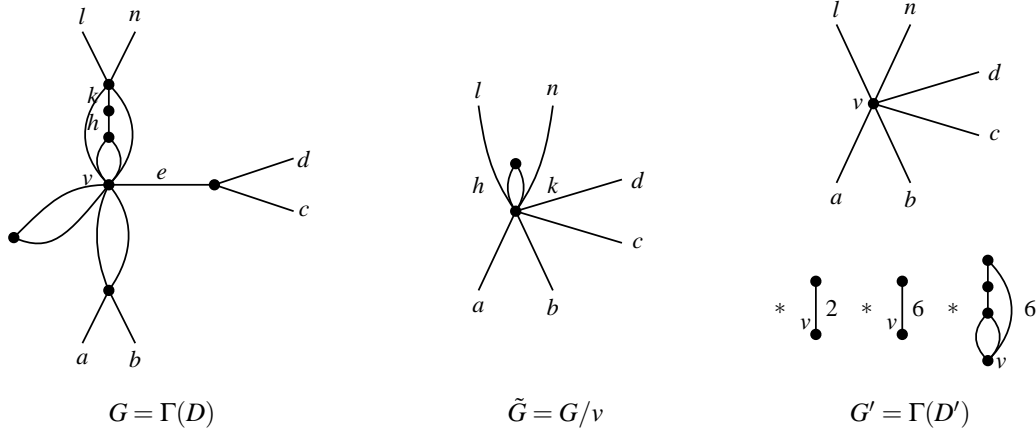
More precisely, the difference between the block component of  $\tilde{G}$  and  $G'$  is that in the last block component  $X$  of  $G'$  in figure 13 the star of  $v$  is contracted to obtain the block of  $\tilde{G}$ . So for the proof of lemma 8.6 in [MP] and (19), we actually need the following lemma.

**Lemma 7.1** If a graph  $H'$  is a contraction of  $H$ , then  $\text{ind}(H') \leq \text{ind}(H)$ .

This lemma can be proved easily by induction on the number of vertices, using the definition of the index. (The main point is that decontraction does not increase edge multiplicity.) *Still it should be understood that the contraction of a vertex is not fully correct as modelling the Murasugi-Przytycki diagram move.*



**Figure 12:** A move of Murasugi-Przytycki, where the relayed strand (dotted line) does not go along a Seifert circle (denoted as  $u$ ) adjacent to  $v$ . The Seifert circles are depicted in gray to indicate that their interior may not be empty.



**Figure 13:** The various Seifert graphs of the diagrams related to the move of Murasugi-Przytycki in figure 12, in the case when the relayed strand does not go along all Seifert circles adjacent to  $v$ . The graph of  $D'$  is given in its block decomposition, which corresponds to the Murasugi sum decomposition along the newly created Seifert circle. For simplicity, we display a multiple edge by attaching the multiplicity to the edge drawn as simple (otherwise, a letter attached just indicates the name).

### 7.3 Modifying the index

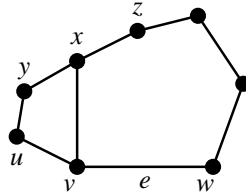
In the following we will need to define several “relatives” of Murasugi-Przytycki’s index. This results from our desire first to fix the aforementioned error, and then to keep track of the braided surfaces. We summarize the various indices in the scheme (71) and the remarks after it. The reader may consult this in advance to avoid confusion.

It becomes necessary to understand exactly the transformation of the Seifert graph  $G$  under the move of Murasugi-Przytycki. We describe it now, also filling in the detail overlooked by them.

Now we use graphs, in which we mark edges (each edge has a  $\mathbb{Z}_2$ -grading: it is either marked or not). In the initial graph all edges are unmarked. A marked edge is to be understood as one that cannot be chosen as an edge  $e$ . It corresponds to a multiple edge. We assume for the rest of the note that  $G$  is *bipartite*, thus  $G$  has no cycles of length 3, which avoids some technical difficulties.

We choose a non-marked edge  $e$  and a vertex  $v$  of  $e$ . Let  $w$  be the other vertex of  $e$  (see figure 12). We define the notion *on the opposite side to  $e$*  as follows.

**Definition 7.1** A vertex  $y \neq v, w$  is on the opposite side to  $e$  if there is a vertex  $x \neq v, w, y$  adjacent to  $v$  such that  $y$  and  $w$  are in different connected components of  $(G \setminus v) \setminus x$ .



(53)

(Here ‘ $\setminus$ ’ stands for the deletion of a vertex together with all its incident edges – but not its adjacent vertices.) Otherwise we say  $y$  is *on the same side as  $e$* .

The meaning of this distinction is that the Murasugi-Przytycki move lays the arc along a Seifert circle  $x$  adjacent to (the Seifert circle of)  $v$ , if  $x$  is on the same side as  $e$ . This move affects the crossings that connect  $x$  to  $v$ , or to a Seifert circle  $z$  on the same side as  $e$ .

**Definition 7.2** We define now the marked graph  $G/e v$ . The vertices of  $G/e v$  are those of  $G$  except  $w$ . The edges and markings on them are chosen by copying those in  $G$  as follows. Let an edge  $e'$  in  $G$  connect vertices  $v_{1,2}$ .

**Case 1.**  $v$  is among  $v_{1,2}$ , say  $v = v_1$ .

**Case 1.1.** If the other vertex  $v_2$  of  $e'$  is  $w$  (i.e.  $e = e'$ ), then  $e'$  is deleted.

**Case 1.2.** If  $v_2$  is on the opposite side to  $e$ , then  $e'$  is retained in  $G/e v$  with the same marking.

**Case 1.3.** If  $v_2$  is on the same side as  $e$ , then  $e'$  is retained in  $G/e v$ , but marked.

**Case 2.**  $v$  is not among  $v_{1,2}$ .

**Case 2.1.** If none of  $v_{1,2}$  is adjacent to  $v$ , then  $e'$  retains in  $G/e v$  the same vertices and marking.

**Case 2.2.** One of  $v_{1,2}$ , say  $v_1$ , is adjacent to  $v$ . (Then  $v_2$  is not adjacent to  $v$  by bipartacy.)

**Case 2.2.1.** If  $v_1 = w$ , then change  $v_1$  to  $v$  in  $G/e v$ , and retain the marking.

**Case 2.2.2.** So assume next  $v_1 \neq w$ . If  $v_2$  is on the opposite side to  $e$ , then retain  $v_{1,2}$  and the marking.

**Case 2.2.3.** If  $v_2$  is on the same side as  $e$ , then we change  $v_1$  to  $v$ , and retain the marking. (Note that by bipartacy, if  $v_2$  is on the same side as  $e$ , then so must be  $v_1$ .)

Since a marking will indicate for us only that the edge cannot be chosen as  $e$ , the resulting graph  $G/e v$  may be reduced by turning a multiple edge into a single marked one. (This also makes it irrelevant to create multiple edges in case 1.3.)

**Theorem 7.1** If  $D'$  arises from  $D$  by a Murasugi-Przytycki diagram move at a crossing  $e$  and Seifert circle  $v$ , then  $\Gamma(D') = \Gamma(D)/e v$ . □

**Definition 7.3** Replace in definition 2.3 the two occurrences of  $G_{v_i} = G/v_i$  by  $G/e v_i$ , as given in definition 7.2. Then we define the corresponding notions of 0-independent edges and index  $\text{ind}_0(D)$ .

With this definition, we obtain (50). The property (51) can be proved by induction over the edge number, looking at a maximal set of independent (not 0-independent) edges, using lemma 7.1 and the observation that  $G/v$  is a contraction of  $G/e v$ . This fixes Murasugi-Przytycki's proof of lemma 8.6 in [MP].

We speculated whether in fact (50) can be stronger than (19). If (52) is false, then conjecture 2.1 is also false, so we were wondering whether this is a way to find counterexamples to that conjecture. We explained, though, that indeed (52) is true (and proved by Traczyk [Tr2] as a followup to our discovery of Murasugi-Przytycki's gap). One could then say that this makes our (much more awkward) definition of index obsolete. Still its treatment is necessary in order to prove (51) or (52), and fix the gap in [MP].

Also, if one likes to keep the correspondence between (Seifert) graph and diagram, one must live with the circumstance that (in general) not all of  $\text{star } v$  is to be contracted. The idea of using vertex contraction (straightforwardly, following [MP]) appeared in at least one further paper, [MT]. We feel this misunderstanding may cause a problem at some point, which was additional motivation for us to provide the present correction.

We should stress that, while our proof of (51) might imply that independent edges (in Murasugi-Przytycki's sense of definition 2.3) are 0-independent (i.e. corresponding to a set of single crossings admitting Seifert circle reducing operations), we *do not know* if the converse is true, despite (52). Thus proofs (like those we give below) relying on  $\text{ind}$  (rather than  $\text{ind}_0$ ) are to some extent non-diagrammatic.

The important difference of  $\text{ind}_0$  to  $\text{ind}$  lies in not affecting edges in case 1.2. The treatment of vertices on the opposite side to  $e$ , the technical detail missed by Murasugi-Przytycki, does not affect the result by (52), yet it creates a lot of calculation overhead (which we experienced in attempts to use the possibly better estimate (50) prior to Traczyk's proof of (52)). Note, however, that it implies the additivity of  $\text{ind}_0$  under block sum in an easier (and much more natural) way than Murasugi-Przytycki's corresponding statement for  $\text{ind}$ .

**Definition 7.4** A marked graph is *not 2-connected* if it has an *unmarked* edge whose deletion disconnects it. If  $G$  is not 2-connected, there is a plane curve intersecting  $G$  in a single, and unmarked, edge. We call such a curve a *separating curve*.

Note that the initial (unmarked Seifert) graph of  $D$  is 2-connected because  $D$  has no nugatory crossings.

**Lemma 7.2** If  $G$  is 2-connected, so is  $G/e$ .

**Proof.** We assume to the contrary that  $G/e$  is not 2-connected. Let  $e'$  be a disconnecting edge. So there is a separating curve  $\gamma$  that intersects  $G/e$  only in  $e'$ . The only edges in  $G/e$  which do not exist in  $G$  are of the type  $vz$  in (53) ( $z$  is a vertex on the same side as  $e$ , adjacent to a vertex  $x$  adjacent to  $v$  in  $G$ ). By definition 7.1,  $vz$  belongs to a cycle, and so cannot disconnect  $G/e$ .

Therefore,  $e'$  persists in  $G$ . It must be unmarked in  $G$ , since the move from  $G$  to  $G/e$  never deletes markings. Thus the curve  $\gamma$  must intersect  $G$  in some other edge. The only edges added in  $G$  when recovering it from  $G/e$  (except that  $e$  is decontracted) are of the form  $xz$  in (53) ( $x$  is a vertex adjacent to  $v$ , and  $z$  is a vertex adjacent to  $x$  on the same side as  $e$ ). Then  $\gamma$  passes in  $G$  through a cycle as the right one in (53) (the one containing  $z, x, v, w$  in consecutive order; note that  $z \neq w$  by bipartacy). In  $G/e$  this cycle is affected only by replacing  $zx, xv$  by  $zv$  (and contracting  $e$ ). So  $\gamma$  must pass through  $e' = zv$  in  $G/e$ . But by construction  $zv$  is marked in  $G/e$ , and  $\gamma$  is not a separating curve, a contradiction.  $\square$

It is easy to see that  $G_1 * G_2$  is 2-connected iff both  $G_1$  and  $G_2$  are so.

**Lemma 7.3** If  $G_{1,2}$  are 2-connected, then  $\text{ind}_0(G_1 * G_2) = \text{ind}_0(G_1) + \text{ind}_0(G_2)$ .

**Proof.** It is enough to see that the contraction procedure of an edge  $e$  in  $G_1$  does not affect edges or markings in  $G_2$ , except possibly the change of vertex at which the block sum  $G_1 * G_2$  is performed.

Let  $v, w$  be the ends of  $e$ , and we consider the building of  $G/e$  for  $G = G_1 * G_2$ . Let  $z$  be the vertex at which the block sum  $G_1 * G_2$  is performed.

If  $z \neq v$  is not adjacent to  $v$ , then nothing is changed in  $G_2$  when building  $G/e$ .

Next assume  $z = v$ . The vertex  $v$  must be adjacent to at least one more vertex  $x \neq w$  in  $G_1$  (else  $G_1$  is not 2-connected or  $e$  is multiple). Then we see with this choice of  $x$  in definition 7.1 that the vertices in  $G_2$  except  $v$  lie on the opposite side to  $e$ . Thus building  $G/e$  does not affect  $G_2$ .

Finally assume  $z \neq v$ , but  $z$  is adjacent to  $v$ . If  $z = w$  is the other end of  $e$ , then in  $G/e$  all edges incident in  $G_2$  to  $w$  are redirected to  $v$  with the same marking, and so  $G_2$  is not affected. If  $z \neq w$ , then choosing  $z$  for  $x$  in definition 7.1, we see that all vertices of  $G_2$  except  $z$  are on the opposite side to  $e$ . Thus none of these edges is affected by building  $G/e$ .  $\square$

## 7.4 Simplified regularization

To speed up the test for genus 4 generators, we make heavy use of Murasugi-Przytycki's work. We recall the inequalities (17) – (19), and the exactness of inequality (20) in alternating diagrams.

The proof of theorem 1.4 will demonstrate the efficiency of proposition 2.1 as a tool in determining the braid index. To give a first glimpse of that capacity, we mention that we have also obtained the following by computer verification of the tables in KnotScape [HT] and Knotilus [FR]:

**Proposition 7.1** If  $K$  is an alternating knot of  $\leq 18$  crossings, then MWF is sharp, except if  $K$  is the (18 crossing) Murasugi-Przytycki knot or its mutant.  $\square$

This fact also shows the depth of the insight Murasugi-Przytycki must have had in picking up exactly these two knots as counterexample candidates!

The first easy lemma gives an upper control on the growth of the braid index under  $\tilde{t}_2'$  twists.

**Lemma 7.4** If  $\tilde{D}$  is obtained from  $D$  by a  $\tilde{t}_2'$ -twist, then  $\text{ind}(\tilde{D}) \geq \text{ind}(D) + 1$ .

**Proof.** Let  $e$  be the edge in the Seifert graph  $\Gamma(D)$  of  $D$  which is bisected twice to obtain the Seifert graph  $\Gamma(\tilde{D})$  of  $\tilde{D}$ . Double bisection means to put two valence-2-vertices  $v_{1,2}$  on  $e$ , dividing it into three edges  $e_{1,2,3}$ . Let  $e_2$  be the (middle) edge connecting  $v_{1,2}$  in  $\Gamma(\tilde{D})$ .

For an independent edge set  $S$  in  $\Gamma(D)$  we construct an independent edge set  $S'$  in  $\Gamma(\tilde{D})$  by keeping  $S \setminus \{e\}$  and including two of  $e_{1,2,3}$  into  $S'$  if  $e \in S$ , and one of these three edges otherwise. This shows  $\text{ind}(\tilde{D}) \geq \text{ind}(D) + 1$ .

(If  $e$  connects a vertex of valence 2, then one easily sees that  $\text{ind}(\tilde{D}) = \text{ind}(D) + 1$ . However, otherwise we cannot exclude the possibility that  $\text{ind}(\tilde{D}) = \text{ind}(D) + 2$ .)  $\square$

**Remark 7.1** Let us stress again that any choice of two resp. one edge(s) among  $e_{1,2,3}$  to include into  $S'$  will do when  $e \in S$  resp.  $e \notin S$ , provided we specify the to-be-contracted vertices correctly. This observation will be important both when  $e \in S$  (for the proof of corollary 7.1) and  $e \notin S$  (for the proof of lemma 7.5).

The next step is to control the degrees of the skein polynomial to estimate the braid index from below.

**Lemma 7.5** Let  $D$  be an alternating diagram with (see (19))

$$\text{mwf}(D) = \text{mpb}(D). \quad (54)$$

Let  $e$  be an edge in the Seifert graph  $\Gamma(D)$ , which is not contained in at least one maximal independent set. (That is, there is a maximal independent set  $C$  with  $e \notin C$ .) Assume further that  $e$  is either a simple edge (i.e. its crossing has no Seifert equivalent one) or that  $D$  is special.

Let  $\tilde{D}$  be obtained from  $D$  by a  $\tilde{t}_2$  twist at (the crossing corresponding to)  $e$ . Then

1. (54) holds for  $\tilde{D}$ ,
2.  $\text{ind}(\tilde{D}) = \text{ind}(D) + 1$  and  $\text{mwf}(\tilde{D}) = \text{mwf}(D) + 1$ , and
3.  $e$  is not contained in some maximal independent set of  $\Gamma(\tilde{D})$ . (Here  $e$  is to be considered as a crossing in  $\tilde{D}$  and identified with some of the three crossings after the twist.)

**Proof.** The equality (54), together with (17) – (19) and the exactness of inequality (20), imply that (17), (18) are exact for  $D$ . Now let  $D'$  be  $D$  with  $e$  switched. Assume w.l.o.g.  $e$  is positive in  $D$  (the negative case is handled analogously).

We want to show first

$$\min \deg_l P(\tilde{D}) = \min \deg_l P(D). \quad (55)$$

If  $D$  is special (alternating), (55) holds, because both hand sides equal  $1 - \chi(D)$ . So assume  $e$  is simple. Then  $\text{ind}_-(D') = \text{ind}_-(D) + 1$ , because in  $\Gamma(D')$  we have a single negative edge in a block. Denote by  $q_1(D)$  and  $q_2(D)$  the right hand-sides of the inequalities (17) and (18). Since  $w(D') = w(D) - 2$ , it follows that  $q_2(D') = q_2(D)$ , and since (18) is exact for  $D$ , we have

$$\min \deg_l P(D') \geq \min \deg_l P(D). \quad (56)$$

Since (by the skein relation)

$$P(\tilde{D}) = (l^2 + 1)P(D) - l^2 P(D'), \quad (57)$$

we see that (55) holds.

Now let us look at (17). For an alternating diagram  $D$ , as remarked, all blocks of  $G = \Gamma(D)$  have edges of the same sign. Thus theorem 2.8 implies that the intersection of each maximal independent set of  $G$  with the positive blocks also realizes  $\text{ind}_+(G)$ . The assumption that  $e$  is not contained in some maximal independent set of  $G$  (and the exactness of (20) in alternating diagrams) means then that  $\text{ind}_+(D') \geq \text{ind}_+(D)$ . Thus  $q_1(D') < q_1(D)$ . Since (17) is exact for  $D$ , we see from (57) that

$$\max \deg_l P(\tilde{D}) = \max \deg_l P(D) + 2. \quad (58)$$

Now, (58) and (55) imply that  $\text{mwf}(\tilde{D}) = \text{mwf}(D) + 1$ , and because  $s(\tilde{D}) = s(D) + 2$  and (54), we have that  $\text{ind}(\tilde{D}) \leq \text{ind}(D) + 1$ . The opposite inequality was shown in lemma 7.4. So  $\text{ind}(\tilde{D}) = \text{ind}(D) + 1$ , and the second claim in our lemma follows. The first claim is then also clear. The third claim is a consequence of remark 7.1.  $\square$

Note that condition 3 implies that one can iterate the twisting. So we have a test for sharpness of MWF on a given series.

**Corollary 7.1** Assume  $D$  is a special alternating generator diagram such that (54) holds, and let  $S$  be the intersection of all maximal independent sets of  $D$ . Let for  $S' \subset S$  the diagram  $D_{S'}$  be obtained from  $D$  by applying one  $\tilde{t}_2$  twist at each crossing in  $S'$ . Assume that for all  $S'$

$$\text{mwf}(D_{S'}) = \text{mwf}(D) + |S'|. \quad (59)$$

Then the MWF inequality is exact on, and conjecture 2.1 is true for the series of  $D$ .

**Proof.** As before, we split the series of  $D$  into  $2^{|S|}$  subseries, depending on whether we twist or not at each crossing in  $S$ . We know that each twist augments the index by at least one. The equalities (54) and (59) then ensure that it goes up by exactly one under a  $\tilde{t}_2$  twist at some choice of crossings  $s$  from  $S$ . So for each of the 3  $\sim$ -equivalent crossings  $s'$  obtained from  $s$  under this twist, one can choose a maximal independent set not to contain  $s'$  by remark 7.1. Then one can apply further  $\tilde{t}_2$  twists at  $s$ , and the sharpness of MWF is preserved.  $\square$

For non-special generators we must take into account Seifert equivalence classes.

**Corollary 7.2** Assume for a non-special alternating generator diagram  $D$  that (54) holds, and let  $S$ ,  $S'$  and  $D_{S'}$  be as in corollary 7.1. Let  $S''$  be a set of crossings of  $D$ , such that the intersection of  $S''$  with each Seifert equivalence class of  $n$  crossings contains at most  $n - 1$  elements. (In particular,  $S''$  is disjoint from trivial Seifert equivalence classes, and so  $S'' \cap S = \emptyset$ .) Assume that for all such sets  $S'$  and  $S''$  we have

$$\text{mwf}(D_{S' \cup S''}) = \text{mwf}(D) + |S'| + |S''|. \quad (60)$$

Then the MWF inequality is exact on, and conjecture 2.1 is true for the series of  $D$ .

**Proof.** The proof is completely analogous. When twisting further at the classes twisted in  $S''$  (resp. the one remaining element in a Seifert equivalence class not in  $S''$ ), all edges are simple.  $\square$

Note the following slight simplification. Each of the Seifert equivalence classes  $X_i$  of  $D$  decomposes by lemma 2.2 completely into  $\sim$ -equivalence classes  $Y_{i,j}$ . If for two sets  $S''$  and  $\tilde{S}''$  we have  $|S'' \cap Y_{i,j}| = |\tilde{S}'' \cap Y_{i,j}|$  for all applicable  $i, j$ , then  $D_{S' \cup S''}$  and  $D_{S' \cup \tilde{S}''}$  differ by a mutation. So we need to choose just one  $S''$  for given tuple of sizes  $\{|S'' \cap Y_{i,j}|\}_{i,j}$ .

**Proof of theorem 1.4.** This proof consists now in a verification that bases heavily on corollaries 7.1 and 7.2. First consider one diagram  $D$  of each generator knot  $K$ .

The decisive merit of these corollaries is that they provide a condition to test sharpness of MWF by calculation of  $P$  only on very few diagrams in the series, and without adding a high number of crossings by  $\tilde{t}_2$  twists. The previous verification needs  $2^{t(D)}$  diagrams, with  $t(D)$  being the number of  $\sim$ -equivalence classes. Note that by theorem 3.1 for  $g(D) = 4$  we have  $c(D) \leq 33$  and  $t(D) \leq 21$ , so with that naive method, we would have up to  $2^{21}$  diagrams per generator, with up to 75 crossings.

The graph theoretic determination of  $S$  is not computationally trivial, but takes only a fraction of the time that would have been required to calculate the large number of extra polynomials. It can be done by essentially the same recursion (and thus takes about as long) as the calculation of the index:

$$S = S_G = \bigcap \left\{ \{e\} \cup S_{G/v} : \begin{array}{l} e \text{ simple edge of } G, v \text{ vertex} \\ \text{of } e, \text{ind}(G/v) = \text{ind}(G) - 1 \end{array} \right\}.$$

(And of course  $S = \emptyset$  if  $G$  has no simple edge.) This calculation reveals (see (61)) that  $S$  is small; always  $|S| \leq 5$ . For most generators  $S$  is actually empty.



For special generators the calculation of the index from the definition is more time consuming, because they have more crossings than the non-special ones, and admit no Murasugi-sum decomposition. This suggested to seek some further simplifications. We found an easy sufficient condition, which we describe in proposition 8.3, to assure that  $S$  is empty. In case of a special generator  $D$  with empty  $S$  only the polynomial of  $D$  needs to be calculated.

Proposition 8.3 is stated and proved in §8, since it relates to some further notions treated there. The generator  $12_{1202}$  of genus two easily shows that the inequality (69) in the proposition is false for non-special generators. On the other hand, (70) holds, and (69) becomes an equality for all special genus two generators. For genus four, it is still exact for slightly more than one half (748,193) of the special generators.

Even when proposition 8.3 does not apply directly, its proof shows some priority that should be given to the contraction of vertices of valence two. Note that the only difficulty for non-special generators is that  $F$  may not be a forest, and we need to take care of not contracting vertices of valence 2 incident to a double edge. With this additional restriction, one can deal with non-special generators, too, and this speeds up considerably their calculation.

The following table shows the sizes of intersections  $S$  of maximal independent sets for special and non-special genus 4 generators; the last column gives the total number of extra, non-generator, knots whose polynomials needed to be calculated. The most complicated such knots have 35 crossings. For a special generator, their number is  $2^{\#S} - 1$ .

#S	0	1	2	3	4	5	total test knots
special	1,082,270	269,406	106,204	20,676	1649	33	758,508
non-spec	1,573,426	290,668	63,821	6,488	178	—	15,331,751

(61)

Even though the combination of  $S'$  with the sets  $S''$  creates extra work for the non-special generators, this overhead was manageable. We used the simplification for the choice of  $S''$  explained after the proof of corollary 7.2. (There are theoretic ways to further reduce the calculation, but they result in too technical conditions, whose implementation augments the risk of an error.) In total, non-special generators required less time to deal with, because the calculation of the sets  $S$  for special generators took about 15 times as long as for the non-special ones.

With this, however, the work is not yet finished. The lack of (at least confirmed) flype invariance of  $\text{ind}(D)$  requires us to deal with diagrams  $D_0$  obtained by flypes from the generator diagrams  $D$  we considered. To simplify the occurring problem, first note that type B flypes (see figure 5) commute with  $\vec{r}'_2$  twists, so it is enough to generate  $D_0$  from  $D$  by type A flypes. (This leads already only to relatively few new diagrams  $D_0$ , about 58000 for special generators  $K$  and about 150000 for non-special ones.) Also, the condition (70) is invariant under flypes, so no  $D_0$  coming from diagrams  $D$  we discarded using proposition 8.3 need to be considered. Moreover, the mutation invariance of  $P$  ensures that the MWF bound will grow (by 1) for each  $\vec{r}'_2$  twist we apply on  $D_0$ . Since we argued that the diagram index will grow also (at least) by 1, the verification we need to perform on the remaining  $D_0$  reduces to the check of (54). (We determined in fact also the intersections  $S$  of maximal independent sets for all the diagrams  $D_0$ , and found that they have the same size as for the corresponding diagrams  $D$  that differ by type A flypes.)

We must at last admit that, even after these various arguments and simplifications, still several weeks were necessary to complete the work.  $\square$

**Remark 7.2** It is clear from our proofs that the conjecture 2.1 of Murasugi-Przytycki is also confirmed for the knots in theorem 1.4 and proposition 7.1.

Using [STV], we have for example:

**Corollary 7.3** The number of alternating genus 2 resp. 3 knots of braid index  $n$  grows like  $O(n^8)$  resp.  $O(n^{14})$ . The number of achiral such knots grows like  $O(n^2)$  resp.  $O(n^5)$ .

**Proof.** Regularity of all generators implies that, up to an additive constant, the braid index behaves like half of the crossing number on such knots. Then the results on enumeration by crossing number extend directly by replacing crossing number by braid index.  $\square$

## 7.5 A conjecture

The above work suggests that for fixed genus,  $b$  should behave similarly to  $c/2$ . A calculation of the range of the difference  $c - 2b$  for alternating knots  $K$  of genus  $g \leq 4$  reveals that we have

$$g - 2 \stackrel{(*)}{\leq} c - 2b \leq 2g - 3 \quad \text{if } K \text{ is special alternating,} \quad (62)$$

$$-2 \leq c - 2b \leq 2g - 4 \quad \text{if } K \text{ is not special alternating,} \quad (63)$$

with all inequalities realized sharply. In fact, both estimates from above are shown in [St13] for all alternating *knots*  $K$  (of all  $g$ ), and we see that they are the best possible (at least for  $g \leq 4$ ). The lower estimate in (63) follows by the result of Ohyaama [Oh]. The improved lower bound  $(*)$  in (62) seems unclear in contrast. It would be implied by the inequality

$$4 \operatorname{ind}(G) + e(G) \geq 3(v(G) - 1), \quad (64)$$

for a (planar bipartite) graph  $G$  with  $e(G)$  edges and  $v(G)$  vertices, and with *an odd number of spanning trees*. The spanning tree condition is necessary, as show simple examples, and its use makes an approach to prove (64) difficult. Still we verified  $(*)$  and (64) for (the Seifert graphs of alternating diagrams of) all special alternating knots of up to 18 crossings and found no counterexample. If it holds, (64) would in fact then imply  $(*)$  more generally for alternating knots all whose Murasugi atoms (of the alternating diagram; see above theorem 2.8) are knots (and none are links).

One could conjecture  $(*)$  also for links:

**Conjecture 7.1** For a special alternating link  $L$  we have  $c(L) - g(L) \geq 2(b(L) - 1)$ .

In the case of links (64) will take its extended form for general (planar bipartite) graphs  $G$

$$4 \operatorname{ind}(G) + e(G) \geq 3v(G) - 2 - n(D), \quad (65)$$

where  $D$  is the (special alternating) diagram with  $\Gamma(D) = G$  and  $n(D)$  the number of its components. (Note that  $G$  has odd number of spanning trees iff  $n(D) = 1$ ; see [MS].) The other 3 inequalities in (62) and (63) hold also for links, but the upper estimates need some change depending on  $n(D)$ .

There is one more important special case in which one can show our conjecture.

**Proposition 7.2** If  $L$  is a special alternating arborescent link (*without* hidden Conway spheres) then  $c(L) - g(L) \geq 2(b(L) - 1)$ .

**Remark 7.3** The hidden Conway spheres of Menasco [Me] refer to the links which are arborescent and alternating (i.e. have an arborescent diagram and an alternating diagram), but not alternatingly arborescent (i.e. do not have a diagram which is *simultaneously* arborescent and alternating). Even though this family of exceptions is relatively small, it exists, and is often carelessly overlooked by authors. See [Th2] for an explanation.

**Proof.** We show (65) for a special alternating arborescent diagram  $D$  and  $G = \Gamma(D)$ . For a special diagram  $D$ , the Seifert graph  $G$  coincides with one of the checkerboard graphs. If  $D$  is arborescent, latter are series-parallel. This means (see e.g. [St17]) that they can be obtained from  $\bullet \text{---} \bullet$  by edge bisections (putting a vertex of valence two on some edge) and doublings (adding a new edge between the same two vertices). So we assume now  $G$  is (bipartite and) series-parallel. We use induction over the number  $e_s(G)$  of simple edges of  $G$ . (Note that multiple edges enter into the count  $e(G)$  with their multiplicity.)

First assume  $e_s(G) = 0$ . Since decreasing by one the multiplicity  $> 2$  of a multiple edge changes  $n(D)$  by  $\pm 1$ , it is enough to deal with the case that all edges of  $G$  are double. Next one can successively delete double edges (which preserves  $n(D)$  and  $\operatorname{ind}(G) = 0$ ), until one has a tree of double edges. Then the inequality (65) we claimed is easy to see (as an equality; with  $D$  being a connected sum of Hopf link diagrams).

Now assume  $e_s(G) > 0$ . We consider a simple edge  $e$  of  $G$  which is the first in a maximal independent sequence.

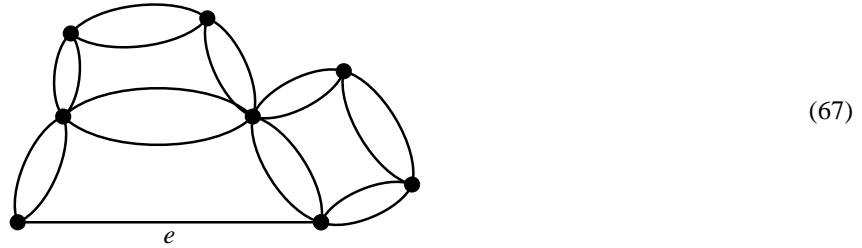
This edge  $e$  can, as a first case, be adjacent to a vertex  $v$  connecting only two vertices



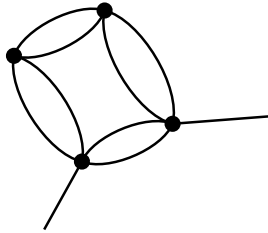
We want to use the definition of index and contract  $v$ .

If  $e'$  is simple, then the contraction preserves  $n(D)$  (it trivializes a reverse clasp), reduces  $v(G)$  by 2 and  $\text{ind}(G)$  at least by 1. The resulting graph is either series-parallel or a block sum of such. So again we can argue by induction and the stability of (65) under connected/block sum. If  $e'$  is multiple, then this can be easily reduced to the argument when  $e'$  is simple. This deals with the case (66).

If we have no fragment (66), then we have a simple edge  $e$  of the following form:




The edges except  $e$  are all multiple, but they can be reduced to double ones using the previous argument. Now from (67) one sees that there is always a cycle in which only two vertices are connected (not necessarily by simple edges, as shown below) to vertices outside the cycle.



By contractions of two double edges like



we can lose 2 vertices, 4 edges, and  $n(D)$  goes down by 2. The index goes not go up by lemma 7.1. It is clear that by such contractions and reducing a multiplicity  $> 3$  of edges by 2, one can turn (67) into . (Keep in mind that the graph is bipartite.) Then  $e$  is no longer simple, and again we can use induction. □

**Corollary 7.4** The inequality  $c(L) - g(L) \geq 2(b(L) - 1)$  holds for special alternating Montesinos links  $L$ .

**Proof.** Such links are arborescent and have no hidden Conway spheres. □

## 8 Minimal string Bennequin surfaces

### 8.1 Statement of result

We apply the work in the previous section to the problem, what knots have a braided surface of minimal genus on the minimal number of strands. With the notation of §2.6, let  $\sigma_{i,k} \in B_n$  for  $1 \leq i < k \leq n$  be the bands we introduced

in (44):

$$\sigma_{i,k} := \sigma_i \dots \sigma_{k-2} \sigma_{k-1} \sigma_{k-2}^{-1} \dots \sigma_i^{-1}.$$

If one represents a link  $L$  as the closure of a braid  $\beta \in B_n$  which is written as a product of  $l$  of the  $\sigma_{i,k}$  and their inverses, one obtains a Seifert surface  $S$  of  $L$  consisting of  $n$  disks and  $l$  bands.  $S$  is called *braided surface* of  $L$ . If  $S$  has minimal genus (i.e. equal to the genus of  $L$ ), then  $S$  is called a *Bennequin surface*. This terminology was coined by Birman and Menasco (see for example [BM]), and relates to the work of Bennequin. He showed in [Be] that such a surface exists for 3-braid links on a 3-string braid. Rudolph [Ru2] showed that *every* (not necessarily minimal genus) Seifert surface can be made into a braided surface. In particular a Bennequin surface always exists on a braid of some (possibly very large) number of strings. A natural question was whether the minimal number of strings (i.e. the braid index of  $L$ ) are always enough to span a Bennequin surface. With M. Hirasawa [HS] we showed later that there are knots of genus 3 and braid index 4, which have no minimal (i.e. 4-)string Bennequin surface. Hirasawa has also shown, contrarily, that for 2-bridge links *any* minimal genus surface is a braided surface on the minimal number of strands. (The same holds for 3-braid links, because in [St13] it was shown that such links have a single minimal genus, in fact even just incompressible, surface.) Here we want to show

**Theorem 8.1** Any alternating knot of genus up to 4 or at most 18 crossings has a minimal string Bennequin surface.

## 8.2 The restricted index

We define now a third variant of graph index, this time one which keeps track of surfaces. As for  $\text{ind}_0$  in §7.3, we use marked edges. In the initial graph  $G = \Gamma(D)$  all edges are unmarked. A marked edge is to be understood as one that cannot be chosen as an edge  $e$ . It corresponds to a crossing that is grouped with other crossings to form a band, or which connects the same 2 Seifert circles as some other crossing. *We assume* again for the rest of the treatment that  $G$  is *bipartite*, and thus has no cycles of length 3, which would create some problems.

First we *reduce* the graph  $G$  by turning a multiple edge into a simple marked one.

Next we choose a non-marked edge  $e$  and a vertex  $v$ . Let  $w$  be the other vertex of  $e$  (see figure 12). Recall the notion on the opposite side to  $e$  from definition 7.1. The meaning of the distinction between the side of  $e$  and the opposite side was that the Murasugi-Przytycki move lays the arc along a Seifert circle  $x$  adjacent to (the Seifert circle of)  $v$ , if  $x$  is on the same side as  $e$ . This move affects the crossings (possibly in bands) that connect  $x$  to  $v$ , or to a Seifert circle  $z$  on the same side as  $e$ .

**Definition 8.1** We define now the marked graph  $G//_e v$ . The vertices of  $G//_e v$  are those of  $G$  except  $w$ . The edges and markings on them are chosen by copying those in  $G$  as follows. Let an edge  $e'$  in  $G$  connect vertices  $v_{1,2}$ .

**Case 1.**  $v$  is among  $v_{1,2}$ , say  $v = v_1$ .

**Case 1.1.** If the other vertex  $v_2$  of  $e'$  is  $w$  (i.e.  $e = e'$ ), then  $e'$  is deleted.

**Case 1.2.** If  $v_2$  is on the opposite side to  $e$ , then  $e'$  is retained in  $G//_e v$  with the same marking.

**Case 1.3.** If  $v_2$  is on the same side as  $e$ , then  $e'$  is retained in  $G//_e v$ , but marked.

**Case 2.**  $v$  is not among  $v_{1,2}$ .

**Case 2.1.** If none of  $v_{1,2}$  is adjacent to  $v$ , then  $e'$  retains in  $G//_e v$  the same vertices and marking.

**Case 2.2.** One of  $v_{1,2}$ , say  $v_1$ , is adjacent to  $v$ . (Then  $v_2$  is not adjacent to  $v$  by bipartacy.)

**Case 2.2.1.** If  $v_1 = w$ , then change  $v_1$  to  $v$  in  $G//_e v$ , and retain the marking.

**Case 2.2.2.** So assume next  $v_1 \neq w$ . If  $v_2$  is on the opposite side to  $e$ , then retain  $v_{1,2}$  and the marking.

**Case 2.2.3.** If  $v_2$  is on the same side as  $e$ , then we change  $v_1$  to  $v$ , and put a marking. (Note that by bipartacy, if  $v_2$  is on the same side as  $e$ , then so must be  $v_1$ .)

Since a marking will indicate for us only that the edge cannot be chosen as  $e$ , the resulting graph  $G//_e v$  may be again reduced by turning a multiple edge into a single marked one. (This also makes it irrelevant to create multiple edges in case 1.3.)

**Definition 8.2** We define the *restricted index*  $\text{ind}_b(G)$  of a graph  $G$  like the (Murasugi-Przytycki) index  $\text{ind}(G)$  in definition 2.3, replacing  $G/v$  by  $G//_e v$ . Again set  $\text{ind}_b(D) = \text{ind}_b(\Gamma(D))$  for a diagram  $D$ .

**Remark 8.1** If in case 2.2.3 of the definition of  $G//_e v$ , we retain the old marking (and do not necessarily put one), then we ignore the restriction coming from keeping bands. So marked edges become the equivalent of multiple ones. Thus the corresponding index is exactly the previously defined  $\text{ind}_0$ , correcting Murasugi-Przytycki's definition of index to reflect their diagram move.

Again we have

**Lemma 8.1** If  $G_{1,2}$  are 2-connected, then  $\text{ind}_b(G_1 * G_2) = \text{ind}_b(G_1) + \text{ind}_b(G_2)$ .

**Proof.** Similar to lemma 7.3. □

**Proposition 8.1** Let  $D$  be a minimal genus diagram of a link  $L$ . (I.e. the canonical surface of  $D$  is a minimal genus surface of  $L$ .) Then  $L$  has a Bennequin surface on a braid of  $s(D) - \text{ind}_b(D)$  strings.

**Proof.** We call a *band* a set of crossings in a diagram which looks locally like a braid of the form

$$\sigma_1 \dots \sigma_{k-1} \sigma_n^{-1} \dots \sigma_{k+1}^{-1} \sigma_k^{\pm 1} \sigma_{k+1} \dots \sigma_n \sigma_{k-1}^{-1} \dots \sigma_1^{-1}$$

for some  $1 \leq k \leq n$ . (This is a fragment of a diagram that contains parts of  $n+1$  Seifert circles.)

A marked edge corresponds to a crossing which either belongs to a band (of  $> 1$  crossing), or which connects the same Seifert circles as at least one other crossing. With this understanding, one has to verify that the definition of  $\text{ind}_b$  using  $G//_e v$  reflects precisely the effect on the diagram by the Murasugi-Przytycki move, keeping track of markings. (Note, however, that a contraction at  $v$  can delete marked edges incident to  $v$ .)

Also, each Murasugi-Przytycki move acts within a single block. So we see that, starting with  $D$ , we can apply *at least*  $\text{ind}_b(D)$  Murasugi-Przytycki moves, with each move reducing the number of Seifert circles and bands by 1. Finally, the moves of Yamada [Ya] can likewise be applied so that the number of (Seifert circles and) bands is preserved. Then we have a braid diagram of  $L$ , which gives the desired surface. With this proposition 8.1 is proved. □

### 8.3 Finding a minimal string Bennequin surface

It is clear now that with proposition 8.1 we have a tool for a computational verification of theorem 8.1. Unfortunately, the description of  $\text{ind}_b$  makes its calculation very time consuming. In practice, we tried to further speed up the recursion. We feel details too technical and too long to present here, but roughly speaking, we tried to take advantage of the observation that handling of vertices 'far apart' in the graph does not depend on their order. We still could not avoid that some sequences of edges were left unverified, so that we may obtain smaller values than  $\text{ind}_b$ . But this computation was much faster, and when its bound already coincides with the one of MWF, we know that the one of  $\text{ind}_b$  must coincide too. This dealt with most generators much more quickly than  $\text{ind}_b$  would have done, and we had to calculate  $\text{ind}_b$  (following the definition) only for the handful of remaining ones.

We explain first the following partial case.

**Proposition 8.2** Alternating knots of genus  $\leq 3$  have a minimal string Bennequin surface.

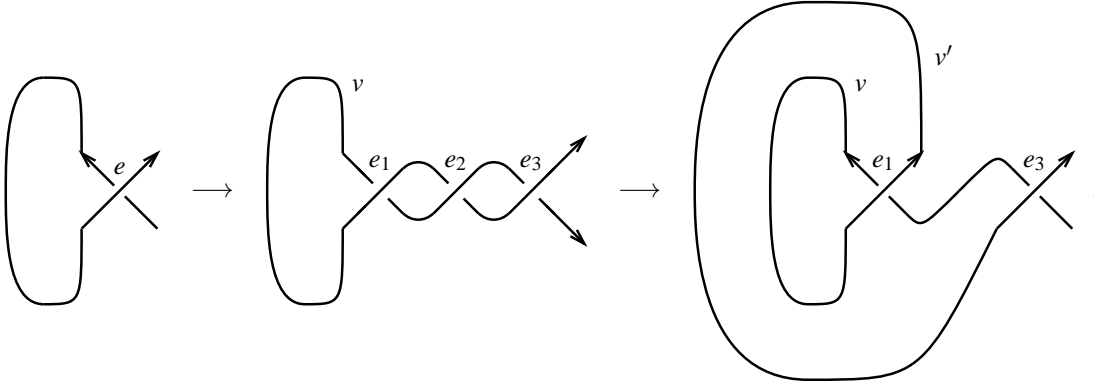
**Proof.** First one checks (by computer) for all generators  $K$  and all their diagrams  $D$  the property

$$\text{ind}(D) = \text{ind}_b(D). \tag{68}$$

It is helpful to distinguish again flypes of type A and type B as in figure 5. Flypes of type B commute with  $\tilde{\tau}_2^2$  twists and type A flypes. So it is enough to generate the diagrams obtainable from a given diagram of a generator  $K$  by

type A flypes. For each such diagram  $D$ , it is enough to find a diagram differing from  $D$  by type B flypes, which satisfies (68). The following argument shows that (68) is preserved under  $\tilde{t}_2^1$  twists.

When (68) holds, then it is easy to see, either by a graph theoretic argument, or by deforming the canonical surface geometrically, that a  $\tilde{t}_2^1$  move at a crossing not deleted by a Murasugi-Przytycki move adds one disk and one band. Consider the case the crossing we twist at is deleted by a Murasugi-Przytycki move. So the edge  $e$  is in a maximal independent set, and we chose a vertex  $v$  to contract. Then we apply first a move



(In the notation we identified edges with crossings and vertices with Seifert circles.) So we alter the middle of the 3 crossings after the  $\tilde{t}_2^1$  move. (We lay the rest of the new Seifert circle  $v'$  below all other crossings attached to  $v$  from outside.) We obtain now at the site of the  $\tilde{t}_2^1$  twist locally a braid  $\sigma_2\sigma_1$ . Then delete by a Murasugi-Przytycki move the edge  $e_3$ , contracting (the vertex of)  $v'$ . So the result follows.  $\square$

The extension to genus 4 requires both more calculation and more delicate implementation techniques of the index. Let us state, however, a simple and useful test that was quoted already in §7.4.

**Proposition 8.3** Assume  $D$  is a special knot generator diagram, and  $t(D)$  is the number of  $\sim$ -equivalence classes of  $D$ . Then

$$\text{mpb}(D) = s(D) - \text{ind}(D) \leq t(D) + \chi(D). \quad (69)$$

Also,  $\text{ind}_b(D) \geq c(D) - t(D)$ . Moreover, the intersection of all independent sets of size  $c(D) - t(D)$  is empty. In particular, if

$$t(D) + \chi(D) = \text{mwf}(D), \quad (70)$$

then MWF is exact and conjecture 2.1 is true on the series of  $D$ .

**Proof.** Up to flypes assume that each non-trivial  $\sim$ -equivalence class of  $D$  forms a clasp.  $D$  has  $c(D)$  crossings and  $c(D) + \chi(D)$  Seifert circles. There are  $c(D) - t(D)$  Seifert circles of valence 2. We claim that for each such Seifert circle we can take one of the edges adjacent to it into an independent set.

Now, because  $D$  is a knot diagram, and is special, the edges incident to valence 2 vertices in the Seifert graph  $\Gamma$  of  $D$  form a graph  $F$ , which is a forest. Moreover, for each region  $R$  of  $\mathbb{R}^2 \setminus \Gamma$  there are at least two edges in  $\Gamma \cap \partial R$  not incident to a valence 2 vertex. (There is at least one such edge because  $D$  is a knot diagram, and then at least one other, because  $\Gamma$  is bipartite.) So one can contract all valence 2 vertices, obtaining  $c(D) + \chi(D) - (c(D) - t(D)) = t(D) + \chi(D)$  vertices.

To find two disjoint independent sets, just choose the opposite edge at each valence 2 vertex. To see that  $\text{ind}_b(D) \geq c(D) - t(D)$ , choose a root  $r$  in each tree of  $F$ , which is not of valence 2. Include in the independent set an edge which is incident to a valence 2 vertex  $v$  adjacent to  $r$ , but not connecting  $r$  and  $v$ . Then contract  $v$  and proceed inductively.  $\square$

**Proof of theorem 8.1.** For genus 4, again we need to test all generator diagrams and all those obtained from them by type A flypes, with the freedom to apply type B flypes on each diagram before calculating  $\text{ind}_b$ .

Some of the special generators to test with the most crossings required up to about half a day of calculation, and this made the work painful. In practice it thus turned out necessary to speed up the procedure.

Some heuristical ways to choose the sequence of vertices  $v$  and edges  $e$  more efficiently were used. Briefly speaking, contracting edges far away from each other commutes, so we need to try only one order of contractions. Accounting for such redundancies made the calculation considerably faster at some places. It still took about 2 weeks to verify (68) for all generators, and thus to complete the work for the genus 4 knots.

In order to give an impression why this simplification is necessary, we mention that much later we tried to recompute the result, for verification purposes, using the definition of  $\text{ind}_b$  as given here. This confirmed the outcome, but took  $3\frac{1}{2}$  months, after splitting the generator list into 100 equal parts and processing them simultaneously. For some generators the calculation took several weeks.

The knots up to 18 crossings served again as a test case and were checked several times with the different index implementations for runtime performance purposes. This concludes the proof of theorem 8.1.  $\square$

**Remark 8.2** There is a question of Rudolph whether a strongly quasipositive knot (or link) always has a strongly quasipositive (Bennequin) surface on the minimal number of strings. While we expect this not to be the case in general, we know of no counterexamples. The theorem shows that the answer is positive for (special) alternating knots of genus up to 4, or up to 18 crossings.

In an attempt at clarification, we summarize the 3 types of index (of *bipartite* graphs or diagrams) that occurred above:

$$\begin{aligned} & \text{ind}_b(D) \\ & \quad \wedge \\ & s(D) - \text{mwf}(D) \stackrel{!}{\geq} \text{ind}_0(D) = \text{ind}(D) \end{aligned} \tag{71}$$

The indices of the first, resp. second, line take, resp do not take, into account keeping the bands. Thus these indices can, resp. can not, be used to estimate Bennequin surface string numbers. The indices of the left, resp. right, side take, resp. do not take, into account the distinction of vertices on the opposite side to an edge. The three indices are arguedly additive under join (theorem 2.8, lemmas 7.3 and 8.1). The inequality marked with a ‘!’ is known to become strict (i.e. not an equality) for some diagrams/graphs. The equality on the right uses Traczyk’s aforementioned argument [Tr2], which we did not present here.

In [St13] we proved the existence of the minimal string Bennequin surface for alternating links of braid index 4. Note that our knot examples in [HS] have braid index 4, crossing number 16 and genus 3. So, if we exclude alternation, they would fall into *any* of the three categories we confirmed the Bennequin surface on. Thus alternation is a crucial assumption. However, the proof of theorem 8.1 and the Murasugi-Przytycki examples [MP, §19] of unsharp MWF inequality ( $5 = \text{mwf} < b = 6$ ) show the difficulties with the extension of the Bennequin surface result for alternating knots both of higher genus and braid index ( $> 4$ ). Still, even if our construction of the Bennequin surface would fail giving minimum number of strings on alternating diagrams (of which we have no example yet), this would by far not mean that there is no minimal string surface. So, with some courage, we could ask:

**Question 8.1** Does every alternating knot have a minimal string Bennequin surface?

Note that our tests are more general than the cases we applied them on. For example, the sharp MWF and minimal string Bennequin surface criteria apply also for an alternating pretzel link  $(x_1, \dots, x_n)$ , with  $x_i$  odd and the twists therein reverse.

Dealing with canonical surfaces leads to some related question: Does even every knot whose genus equals the canonical genus have a minimal string Bennequin surface? It is interesting to remark that for none of the examples reported in [HS] these genera are equal.

## 9 The Alexander polynomial of alternating knots

While it is very well known what Alexander polynomials occur for an arbitrary knot, the question about those occurring for an alternating knot is much harder. We will apply our work to this problem in the last section.

### 9.1 Hoste's conjecture

There is a conjecture I learned from personal communication with Murasugi, who attributes it to Hoste.

**Conjecture 9.1** (Hoste) If  $z \in \mathbb{C}$  is a root of the Alexander polynomial of an alternating knot, then  $\Re z > -1$ .

Here we will use again our generator classification and an appropriate calculation to show

**Theorem 9.1** Hoste's conjecture is true for knots up to genus 4 (or equivalently for Alexander polynomials of maximal degree up to 4).

The parenthetic rephrasing refers to the normalization of  $\Delta$  with  $\Delta(t) = \Delta(1/t)$  and  $\Delta(1) = 1$ . (Beware that we will change this normalization during the proof.)

Some properties of the Alexander polynomials of alternating knots are known; see [Cw, Mu2, MS]. In particular the following holds:

**Theorem 9.2** ([Cw, Mu2]) The coefficients of  $\Delta_K$  for an alternating knot  $K$  alternate in sign, i.e.  $[\Delta_K]_i [\Delta_K]_{i+1} \leq 0$  for all  $i \in \mathbb{Z}$ .

The zeros seem less easy to control than the coefficients, and Hoste's conjecture appears open even for example for 2-bridge knots. Still theorem 9.2 will be a useful ingredient in the proof of theorem 9.1. Also, there is one important tool to control some zeros of  $\Delta$ , the signature.

**Theorem 9.3** For any knot  $K$  the number of zeros of  $\Delta_K$  in  $\{z \in \mathbb{C} : |z| = 1, \Im m z > 0\}$ , counted with multiplicity, is at least  $\frac{1}{2} |\sigma(K)|$ .

This is a folklore fact; see [Ga, St16] for some explanation. In [St16] it was proved alternatively for special alternating knots, and we deduced:

**Corollary 9.1** ([St16]) If  $K$  is special alternating, then all zeros of  $\Delta$  lie on  $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ , and so Hoste's conjecture holds for  $K$ .

(The value  $\Delta(-1)$  is the determinant, and non-zero for example by theorem 9.2; but in fact it is never zero for any knot.)

One further important tool, Rouché's Theorem, will be introduced during the proof.

**Proof of theorem 9.1.** Assume  $\Delta(z) = 0$  for  $\Re z \leq -1$ . Since by theorem 9.2,  $\Delta$  has no zeros on the negative real line, and  $\Delta$  is real and reciprocal, we see that  $\Delta$  must have the 4 distinct zeros  $z^{\pm 1}, \bar{z}^{\pm 1}$ . Then by theorem 9.3, we see that Hoste's conjecture is true whenever  $|\sigma| \geq 2g - 2$ , and in particular the case  $g = 1$  is finished.

Consider next  $g = 2, \sigma = 0$ . Let  $\Delta_{[i]} = [\Delta]_{\min \deg \Delta + i}$ . Then we have

$$\sum_{\Delta(z)=0} z = -\frac{\Delta_{[1]}}{\Delta_{[0]}} \geq 0. \quad (72)$$

However, if  $\Re z \leq -1$ , then the same holds for  $\bar{z}$ , and also  $\Re z^{-1}, \Re \bar{z}^{-1}$  are negative, so that the left sum in (72) is in fact  $< -2$ , a contradiction. This finishes  $g = 2$ .

The argument using (72) works also for  $g = 3, \sigma = 2$ , since the extra pair of zeros on  $S^1$  augments the sum on the left of (72) by at most two, and so it is still negative.

For  $g = 3$  it remains to consider  $\sigma = 0$ , the first non-trivial case.



Note that  $\sigma$  is constant on each series of alternating diagrams (as can be inferred from the combinatorial formula of  $\sigma$ , given for example in [Kf]). Thus we need to consider the series of  $\sigma = 0$  generators. Composite knots (inductively) and mutations are immaterial from the point of view of Hoste's conjecture, so again it suffices to consider prime generators, and one diagram per generator.

Among the 4017 generators of genus 3, only 210 have  $\sigma = 0$ . Now note (e.g. using (72)) that if  $\Delta(z) = 0$ ,  $\Re z \leq -1$ , then the remaining two zeros of  $\Delta$ , different from  $z^{\pm 1}$  and their conjugates, must be real positive (and mutually inverse). Let  $z'$  be the (real) zero  $> 1$ .

Now for any diagram  $D$  in the series  $\langle D' \rangle$  of  $D'$  (resuming the notation of definition 2.9), the Alexander polynomial has the form

$$\Delta(D) = \sum_{i=1}^n a_i \Delta(D_i) \cdot (1-t)^{n_i}, \quad (73)$$

where  $a_i \in \mathbb{N}$ , and  $D_i$  are obtained by smoothing out one crossing in  $n_i$  different  $\sim$ -equivalence classes of  $D'$ . Hereby the convention for  $\Delta$  was *changed*, so that  $\min \deg \Delta = 0$  and  $\Delta_{[0]} = \Delta(0) > 0$ . An explanation of formula (73) is given [SSW]. It was clarified that all terms have leading coefficients of equal sign (and then the same applies to all other coefficients using theorem 9.2).

The polynomials

$$\Delta_i := \Delta(D_i) \cdot (1-t)^{n_i} \quad \text{and} \quad \tilde{\Delta}_i := \Delta(D_i) \quad (74)$$

will be the central object of attention (and calculation) from now on. Note also that

$$n_i = \max \deg \Delta(D') - \max \deg \Delta(D_i) = 2g - \max \deg \Delta(D_i) = 6 - \max \deg \tilde{\Delta}_i. \quad (75)$$

In particular,  $\Delta_i = 0$  if  $n_i > 2g (= 6)$ , and these  $D_i$  and  $\Delta_i$  can be discarded in the sum of (73). (We will soon see that it is very helpful to choose the number  $n$  of terms in this sum as small as possible.)

With this reduction, it is easy to see that

$$\frac{m_1}{M_1} := \inf_{\sup} \left\{ \left| \frac{\Delta_{[1]}(D)}{\Delta_{[0]}(D)} \right| : D \in \langle D' \rangle \right\} = \min_i \gamma(D_i), \quad (76)$$

where

$$\gamma(D_i) := \frac{(\Delta_i)_{[1]}}{(\Delta_i)_{[0]}} = \frac{(\tilde{\Delta}_i)_{[1]}}{(\tilde{\Delta}_i)_{[0]}} + 6 - \max \deg \tilde{\Delta}_i. \quad (77)$$

Then (72) implies that if there is a  $z$  of  $\Re z \leq -1$  with  $\Delta_D(z) = 0$ , then the real zero  $z' > 1$  of  $\Delta_D$  satisfies

$$z' + \frac{1}{z'} > 2 + m_1. \quad (78)$$

Since  $z'$  is the unique zero  $> 1$  (and because  $2 + m_1 \geq 2$ ), (78) is equivalent to

$$\Delta(D)(z_0) < 0, \quad \text{where} \quad z_0 + \frac{1}{z_0} = 2 + m_1 \text{ and } z_0 > 1. \quad (79)$$

(Keep in mind that we normalized  $\Delta$  so that  $\Delta(0) > 0$ , so the leading coefficient is also positive.) Now if we check that contrarily

$$\Delta(D_i)(z_0) > 0 \quad (80)$$

for all  $i = 1, \dots, n$ , then (79) clearly cannot hold. Let us call (80) the *positive zero test*.

For each generator  $D'$  the  $D_i$  can be generated,  $m_1$  and  $z_0$  calculated and then (80) checked. (We excluded  $n_i \geq 7$ , and if  $n_i = 6$  the only polynomial coming in question is  $\tilde{\Delta}_i = 1$ , so we can assume  $n_i \leq 5$ .)

The positive zero test took a few minutes, and succeeded on the 210 generators. With this  $g = 3$  is also finished.

For  $g = 4$  more work is needed. As before  $\sigma \geq 6$  is trivial, but for  $\sigma = 4$  we have something to check. Now with two pairs of (conjugate) zeros of  $\Delta$  on  $S^1$ , (72) and (76) give the condition

$$\left| \frac{\Delta_{[1]}(D)}{\Delta_{[0]}(D)} \right| < 2. \quad (81)$$

To see that this is false, by theorem 9.2 and (75) it is enough to consider for each generator diagram  $D'$  the diagrams  $D_i$  with  $n_i \leq 1$ , and calculate that  $\gamma(D_i) \geq 2$  for  $\gamma(D_i)$  in (77). There are about 500,000 generators of  $g = 4$  and  $\sigma = 4$ , and this test took a few hours, but succeeded.

For  $g = 4$ ,  $\sigma = 2$  we have a modification of the positive zero test used for  $g = 3$ ,  $\sigma = 0$ . Most thoughts apply still, except that in (75) and (77) the '6' (standing for  $2g$ ) becomes '8', and in (78) and (79) the additive 2 on the right disappears, since again we have a new pair of zeros  $z$  on  $S^1$ , giving an additional contribution  $2\Re z$  up to 2 in the sum of (72). (Now we must also take care that  $m_1 \geq 2$ , so that  $z_0$  in (79) is still real.)

The positive zero test thus now weakens (i.e. the condition tested is stronger and may more likely not hold), and there are more generators (about 190,000). There turn up to be 1157 cases of failure. In an attempt to handle these generators  $D'$ , we enhanced the positive zero test.

Apply the test on the diagrams obtained from  $D'$  by  $\bar{t}_2'$  twisting once (separately) at each  $\sim$ -equivalence class of  $D'$ . If the test succeeds on  $D'_i$ , obtained from  $D'$  by a  $\bar{t}_2'$  move at the  $i$ -th class, then we can exclude twists in  $D'$  at this class. With restrictions thus obtained we return again to the positive zero test of  $D'$ . The fewer classes to twist at mean that the set of diagrams  $D_i$  in (73) becomes smaller. Then  $m_1$  in (76), and with also  $z_0$  in (80), goes up, while there are fewer positivity tests to perform.

With this trick from the 1157 cases the number of difficult ones was reduced to 304. These, as well as the  $\sigma = 0$  knots, require a new method, which we sought for a while, and found in Rouché's Theorem.

**Theorem 9.4** (Rouché's Theorem; see e.g. [RS, §1]) If two holomorphic functions  $f, g$  inside and on some piecewise smooth closed contour  $C \subset \mathbb{C}$  satisfy

$$0 < |g(z)| < |f(z)| \quad \text{for all } z \in C, \quad (82)$$

then  $f$  and  $f + g$  have the same number of zeros (with multiplicity) inside  $C$ .

For us  $f, g$  will be always sums of the type (73). We also would like to choose

$$C = \{ \Re z = -1 \}. \quad (83)$$

Now, the contour is not closed. So let us argue why this choice is admissible.

In our case  $f, g$  will be polynomials of the same degree with real leading coefficients  $\max cf f$  and  $\max cf g$ , which do not cancel out. Moreover, all ratios between coefficients are bounded, in a way depending only on the generator  $D'$ . Thus it is not hard to find a constant  $R$  (depending only on  $D'$ ), such that the norms of  $f, g$  compare uniformly on  $\{ \Re z \leq -1, |z| = R \}$  in the same way as their absolute leading coefficients  $|\max cf f|$  and  $|\max cf g|$  compare. If  $\max cf f \neq \max cf g$ , we can achieve (82) after possibly swapping  $f, g$ . If  $\max cf f = \max cf g$ , we take  $g$  in (82) to be our present  $g/2$ , and apply the theorem twice.

With this understanding we assume that  $C$  is chosen as in (83).

We will consider the property that

$$\Re(f\bar{g}) \text{ is positive on all of } C. \quad (84)$$

This positivity condition is insensitive for the tricky case we decided to divide  $g$  by 2. If (84) is satisfied, Rouché's Theorem implies that  $f + g$  has the same number of roots  $z$  with  $\Re z \leq -1$  as *either* of  $f$  or  $g$  do. For  $f$  this is precisely theorem 9.4, assuming with the preceding argument that (82) holds. For  $g$  one applies that theorem on  $f + g$  and  $-g$ , and uses (84) to ensure (82). (Clearly,  $z \in C$  is not a problem in either case, because of (82).)

Therefore, to confirm Hoste's conjecture on  $\langle D' \rangle$ , it suffices that (1) the constant polynomial 1 is among the  $\tilde{\Delta}_i$ , and that (2) all conditions

$$\Re(\Delta_i(z)\Delta_j(\bar{z})) > 0 \quad \text{for } z \in C \text{ and all } 1 \leq i < j \leq n,$$

are satisfied, with  $\Delta_i$  as in (74). Then we can conclude the property that the number of roots  $z$  left of  $C$  is 0 by induction on, say,  $\Delta_{[0]}$ . We call the above two conditions in the following the *Rouché test*.

Now the number of checks grows quadratically in  $n$ , which makes the reduction of the number of  $\Delta_i$  much more urgent than before. We already discarded zero polynomials. Clearly, duplicate  $\Delta_i$ , even up to scalars, can be

discarded too, and one easily sees that in fact this is true also up to multiplication by powers of  $1 - t$ . So we divided all factors  $1 - t$  out of the  $\Delta_i$  prior to discarding duplicates. Since the origin of the values  $n_i$  in (74) will be irrelevant for the rest of the calculation, assume w.l.o.g. that  $\tilde{\Delta}_i$  are the so reduced polynomials (indivisible by  $t - 1$ ). By symmetry one sees then that  $\max \deg \tilde{\Delta}_i$  is an even number (between 0 and 8).

For the purpose of testing the 304 remaining  $\sigma = 2$  generators, we used MATHEMATICA™ [Wo] and the Rouché test with the so far reduced sets of  $\tilde{\Delta}_i$  for each  $D'$ . In determining the  $\tilde{\Delta}_i$ , we also used the restrictions, obtained in the enhanced positive zero test, on which  $\sim$ -equivalence classes of the generator twists are necessary. After an hour and 15 minutes MATHEMATICA reported success on all 304 generators  $D'$ , thus completing the  $\sigma = 2$  case.

For  $\sigma = 0$ , however, there are about 60,000 generators (with no restrictions on  $\sim$ -equivalence classes to twist at), and MATHEMATICA™, albeit very intelligent, is known to pay for its intelligence with its speed. This made further reductions of the set of  $\tilde{\Delta}_i$  necessary.

In this vein, note that if some  $\tilde{\Delta}_i$  lies in the convex hull of others, it is also redundant. (It is enough that  $a_i$  remain positive in (73); integrality is inessential.) Unfortunately, the effort of identifying exactly convex linear combinations as a pre-reduction to the Rouché test would take not less effort than the test itself.

In practice we did the following for polynomials  $\tilde{\Delta}_i$  of degrees  $d = 2, 4$ . We determined the minimal and maximal ratios of  $(\tilde{\Delta}_i)_{[j]} / (\tilde{\Delta}_i)_{[0]}$  for  $1 \leq j \leq d/2$ . Then we replaced the set of all  $\tilde{\Delta}_i$  of degree  $d$  by a collection of  $2^{d/2}$  polynomials given by taking for each coefficient  $\Delta_{[j]}$  with  $1 \leq j \leq d/2$  once the minimal, and once the maximal ratio, setting  $\Delta_{[0]} = 1$  (and completing the other coefficients by symmetry  $\Delta_{[d-j]} = \Delta_{[j]}$ ).

For  $d = 2$  this replacement is equivalent (under convex linear combinations), so it is a genuine simplification, but this did not speed up the check sufficiently. Contrarily, for  $d = 4$  the new polynomials have a larger convex hull, which augments the risk of failure of the test. However, using only the 4 polynomials instead of the dozens of others made the test about 10 times faster. It still took a couple of days to carry out.

MATHEMATICA reported success of the last, fastest, form of the test on all  $\sigma = 0$  generators, except only one, the knot  $12_{1039}$  of [HT]. There the test worked out at least using the full list of polynomials  $\tilde{\Delta}_i$  of degree 4, rather than their 4 substitutes.

With this, after a few weeks of work, the check of Hoste's conjecture in genus  $\leq 4$  was completed.  $\square$

**Remark 9.1** After we finished our proof, we learned of Ozsváth and Szabó's inequality (88). It would replace the r.h.s. of (72) by 2, thereby settling  $g = 3$ ,  $\sigma = 0$  directly, and sharpening the positive zero test for  $g = 4$ . Also, (81) would be excluded straightforwardly. Still both the nature of the tests and the extent of calculation would not be reduced significantly.

Note that (76) can be extended easily to  $|\Delta_{[i]} / \Delta_{[0]}|$  for  $i > 1$ . In particular we have

**Proposition 9.1** For given  $i \geq 1$  and  $g > 0$ , the ratios  $\{ |\Delta_{[i]} / \Delta_{[0]}| : g(K) = g \}$  are bounded.  $\square$

Such a property is not at all evident without using generators. Of course, it is possible, using e.g. corollary 3.1, to give explicit estimates in terms of  $g$ . Moreover, we gain a practical method to calculate (even sharp) upper and lower bounds on this ratio for given genus when the generators are available.

**Example 9.1** For example, for  $g = 4$ ,  $\sigma = 0$  prime alternating knots we found

$$3 \leq -\frac{\Delta_{[1]}}{\Delta_{[0]}} \leq 20, \quad 5 \leq \frac{\Delta_{[2]}}{\Delta_{[0]}} \leq 122, \quad 7 \leq -\frac{\Delta_{[3]}}{\Delta_{[0]}} \leq 333, \quad \text{and} \quad 8 \leq \frac{\Delta_{[4]}}{\Delta_{[0]}} \leq 461,$$

and these bounds are the best possible (except I did not check if equalities are attained, or the inequalities are always strict). They were found by unifying the analogous bounds that can be obtained for each generator separately. Using latter, we also estimated for each generator the Jensen integral for the Euclidean Mahler measure  $M(\Delta)$ , the product of norms of zeros of  $\Delta$  outside the unit circle (see [SSW]). In combination, we have

$$M(\Delta) \leq 638.21$$

for the Euclidean Mahler measure of  $g = 4$ ,  $\sigma = 0$  prime alternating knots. (This estimate may not be very sharp.)

Such data were obtained in an earlier attempt at this case of Hoste's conjecture, prior to using Rouché's theorem. They can be extended with the proper (not small, but manageable) amount of calculation to all  $g = 4$  alternating knots. Note again that the special alternating knots (here  $\sigma = 8$ ) are mostly clear from this point of view. Even for arbitrary genus  $g$  it is easy to deduce from the preceding discussion that

$$0 \leq (-1)^j \frac{\Delta_{[j]}}{\Delta_{[0]}} < \binom{2g}{j},$$

with the right inequality the best possible and the left bound not improvable beyond 1. (The bound is exactly 1 for  $j = 1$  by the work in [MS]; apparently this is true also for higher  $j$ , and would follow from Fox's Trapezoidal Conjecture, discussed in [HS, St16] and right below.) Also  $M(\Delta) \equiv 1$ .

By modifying the contour  $C$  in Rouché's test one could obtain further more specific information about location of  $\Delta$ -zeros on certain families of knots.

## 9.2 The log-concavity conjecture

The last open problem we consider is the log-concavity conjecture. This conjecture, made in [St16], states:

**Conjecture 9.2** Call a polynomial  $X \in \mathbb{Z}[t^{\pm 1}]$  to be *log-concave*, if  $X_{[k]} := [X]_k$  are log-concave, i.e.

$$X_{[k]}^2 \geq X_{[k+1]}X_{[k-1]} \quad (85)$$

for all  $k \in \mathbb{Z}$ . Let  $n(K)$  be the number of components of a link  $K$ .

- (1) If  $K$  is an alternating link, then  $t^{(1-n(K))/2} \Delta_K(t)$  is log-concave.
- (2) If  $K$  is a positive link, then  $t^{(1-n(K))/2} \nabla_K(\sqrt{t})$  is log-concave.

We will refer to both properties as 'Δ-log-concavity' resp. '∇-log-concavity'.

We remind that by Crowell-Murasugi [Cw, Mu2] for part 1, when  $\Delta_K(t)$  is normalized so that  $\min \deg \Delta_K = 0$  and  $[\Delta_K]_0 > 0$ , the polynomial  $\Delta_K(-t)$  is positive, i.e. all its coefficients are non-negative. The same property holds by Cromwell [Cr] for part 2 and  $t^{(1-n(K))/2} \nabla_K(\sqrt{t})$ .

Part 1 is a natural strengthening of Fox's Trapezoidal conjecture.

**Conjecture 9.3** (Fox) If  $K$  is an alternating knot and  $\Delta_K$  is normalized so that  $\min \deg \Delta_K = 0$ , then there is a number  $0 \leq n \leq g(K)$  such that for  $\Delta_{[k]} := [\Delta_K]_k$  we have

$$\begin{aligned} \Delta_{[k]} &> \Delta_{[k-1]} && \text{for } k = 1, \dots, n, \\ \Delta_{[k]} &= \Delta_{[k-1]} && \text{for } k = n+1, \dots, g(K). \end{aligned} \quad (86)$$

We call polynomials of this form *trapezoidal*. A similar property can be conjectured for (non-split) alternating links, replacing  $g(K)$  by  $\lfloor \text{span } \Delta_K / 2 \rfloor$ .

The Trapezoidal conjecture has received some treatment in the literature, being verified for 2-bridge knots [Ha] (see also [Bu]) and later for a larger class of alternating algebraic knots [Mu6]. More recently, Ozsváth and Szabó used their knot Floer homology [OS] to derive a family of linear inequalities on the coefficients of  $\Delta$  for an alternating knot.

**Proposition 9.2** (Ozsváth and Szabó) Let  $K$  be an alternating knot of signature  $\sigma = \sigma(K)$ , and genus  $g = g(K)$ , and let  $\Delta = \Delta(K)$  be normalized so that  $\min \deg \Delta = 0$  and  $[\Delta]_0 > 0$ . Then for each integer  $s \geq 0$ ,

$$(-1)^{s+g} \sum_{j=1}^{g-s} j \cdot [\Delta]_{s+g+j} \leq (-1)^{s+\sigma/2} \max \left( 0, \left\lceil \frac{|\sigma| - 2s}{4} \right\rceil \right). \quad (87)$$

(Note that when  $\Delta$  is normalized so that  $\Delta(1) = 1$  and  $\Delta(t) = \Delta(1/t)$ , then the sign of  $[\Delta]_{\pm g}$  is  $(-1)^{g+\sigma/2}$ .)

For genus  $g = 2$  the inequalities (87) are very similar to (and slightly stronger than) the Trapezoidal conjecture, and for  $s = g - 2$  yield

$$-[\Delta]_1 \geq 2[\Delta]_0 + \begin{cases} -1 & \text{if } |\sigma| = 2g \\ +1 & \text{if } |\sigma| = 2g - 2 \\ 0 & \text{otherwise} \end{cases}, \quad (88)$$

which settles in (86) the case  $k = 1$  (for knots). For  $g > 2$  and  $s < g - 2$ , the inequalities (87) do not relate directly to the Trapezoidal conjecture.

In-Dae Jong [Jn] has proved independently the Trapezoidal conjecture up to genus 2 using the generator description in theorem 2.12, and observed that for genus 2 the log-concavity of  $\Delta$  easily follows from trapezoidality.

In [St16] we showed part 2 of conjecture 9.2 for special alternating knots, by giving a new proof of theorem 9.3 in this case.

We have now

**Theorem 9.5** Both part 1 and 2 of conjecture 9.2 (and therefore also Fox's conjecture) hold for knots of genus at most 4.

The preceding work easily settles the positive case.

**Proof of part 2 of theorem 9.5.** From theorem 9.3 one easily sees that all zeros of  $\nabla(\sqrt{t})$  are real if  $\sigma(K) \geq 2g(K) - 2$ . With the explanation in [St16] we have then log-concavity of  $\nabla$ . Thus we need to check just the polynomial of  $14_{45657}$  directly.  $\square$

Part 1 is much harder to check, since we have no easy sufficient conditions at hand. We explain how we proceeded.

**Proof of part 1 of theorem 9.5.** We had to test all generators up to genus 4. A small defect of log-concavity is that it is not straightforwardly preserved under product (as is trapezoidality), so composite generators must be handled either.

Due to its volume, the check had to be optimized strongly. Consider a particular generator diagram  $D$ . We use the notation of (74) in the proof of theorem 9.1. Since one can recover  $\Delta_i$  by

$$\Delta_i = (1 - t)^{g(D) - \max \deg \tilde{\Delta}_i / 2} \tilde{\Delta}_i \quad (89)$$

even when  $\tilde{\Delta}_i$  is taken up to powers of  $1 - t$ , we can assume w.l.o.g. that we divide out all factors  $1 - t$  in  $\tilde{\Delta}_i$  so that  $\tilde{\Delta}_i(1) \neq 0$ , and  $[\tilde{\Delta}_i]_0 > 0$ . First we designed the calculation of  $\tilde{\Delta}_i$  so that diagrams  $D_i$  of genus 0 or split ones (where  $\tilde{\Delta}_i$  is zero or a scalar) are detected and discarded in advance. For the calculated  $\tilde{\Delta}_i$  again those of small degree are treated extra. For degree 0 it is enough to keep the polynomial 1. For degree 2 we need only two polynomials, those whose ratio  $[\tilde{\Delta}_i]_1 / [\tilde{\Delta}_i]_0$  is minimal and maximal.

The  $\tilde{\Delta}_i$  of degree 4 lead to the main reduction. In this case we have a number of points

$$([\tilde{\Delta}_i]_1 / [\tilde{\Delta}_i]_0, [\tilde{\Delta}_i]_2 / [\tilde{\Delta}_i]_0) \in \mathbb{R}^2 \quad (90)$$

in the plane. We need to keep only the *extremal* ones, i.e. those which span the convex hull, and discard the others (which lie in the convex hull). In practice it turned out that this reduced the number of  $\tilde{\Delta}_i$  of degree 4 up to a factor of 150, and the number of all  $\tilde{\Delta}_i$  up to a factor of 11.5.

With the so reduced set of  $\tilde{\Delta}_i$ , we recover  $\Delta_i$  by (89) and must test for  $1 \leq k \leq g(D)$

$$[\Delta_i]_k^2 \geq [\Delta_i]_{k-1} [\Delta_i]_{k+1}, \quad (91)$$

and the set of such conditions for convex linear combinations of  $\Delta_i$  and  $k < g(D)$ . (The case  $k = g(D)$  for linear combinations follows directly from (91) and symmetry of  $\Delta_i$ .) Latter again turns into deciding whether the convex polygon  $\Sigma \subset \mathbb{R}^2$  spanned by points

$$(-[\Delta_i]_{k+1} / [\Delta_i]_k, [\Delta_i]_{k+2} / [\Delta_i]_k) \in \mathbb{R}^2$$

is contained in the region  $R = \{(x, y) : x \geq 0, y \leq x^2\}$ . Clearly it is enough to test the inclusion  $\partial\Sigma \subset R$  for the boundary  $\partial\Sigma$  of  $\Sigma$ . This boundary is a certain collection of edges between extremal points, and can be determined similarly to the convex hull reduction of the  $\tilde{\Delta}_i$  of degree 4. Finally, we have to check that the function  $\mathbb{R}^2 \ni (x, y) \mapsto x^2 - y$  is non-negative on these edges. In practice, we made the tests by an  $\varepsilon$  stricter, to avoid floating point number deviations.

For a handful of pairs  $(D, k)$  we have  $\Sigma \not\subset R$ . Let us call such pairs and the corresponding generators  $D$  *irregular*. For genus 3 the irregular generators are  $7_1$  and  $8_5$ . They still do not mean that we have a counterexample, since the points coming from  $\Delta$  polynomials of diagrams in the series of  $D$  form only a subset of  $\Sigma$ . In such cases again we check log-concavity of the generator polynomial  $\Delta(D)$ , and then apply the described test to diagrams  $D'$  obtained by one  $\tilde{r}_2^+$  twist from  $D$ . Iterating this a few times shows that only one explicit one-parameter family, of  $(3, 3, 2m)$  pretzel diagrams ( $m > 0$ ), remains, which can be verified directly.

Among prime generators of genus 4, only 3 are irregular, and they are  $9_1$ ,  $10_2$  and  $10_{46}$ . Using the twisted diagram check, they leave out only one one-parameter family each, consisting of the rational knots  $2m - 1, 8$  and the pretzel knots  $P(2m, 1, 7)$  and  $P(2m, 3, 5)$  (with  $m > 0$ ) resp. Again latter are easily settled directly. Among composite generators up to genus 4, no irregular ones occur.

Despite its theoretical simplicity, the check involved much skillful implementational effort. Among others, we decided to write an own (optimized) procedure for calculating the Alexander polynomial, instead of using the functionality of [HT], which applies a substitution to the skein polynomial obtained by the Millett-Ewing algorithm.

With all optimization, we had a speed-up by a factor of almost 5 in comparison to the initial implementation. Finally, while genus 3 could be done in about 8 minutes, we could reduce the effort for genus 4 only to about a second for the simplest generators, and up to 30 seconds for the most complicated ones. This meant that the verification of the full list of generators had to be done in a few dozen parts, a couple of them running over several weeks.  $\square$

**Remark 9.2** Note that for Fox's conjecture the test could be simplified, since dealing with linear combinations of the  $\Delta_i$  becomes obsolete. (The difficulty of log-concavity is that (85) is not a linear condition.) Still this would not have reduced the magnitude of calculation significantly, since the most time is required for the calculation of the  $\Delta_i$ .

With some effort to adapt the computation to links, we obtained the following outcome.

**Theorem 9.6** The  $\Delta$ -log-concavity (and hence Fox's conjecture 9.3) holds for  $n$ -component alternating links  $L$  with  $\text{span} \Delta(L) + n \leq 9$  (or equivalently  $g(L) + n \leq 5$ ).

**Proof.** We recurred the calculation from generators of these links to knot generators by using clasplings (24). The check turned out far more laborious than for knots, but there is no real difference in the method, so we skip details to save space.  $\square$

**Remark 9.3** It is suspectable that in conjecture 9.3 we always have for a knot  $K$

$$n \geq g(K) - |\sigma(K)/2|. \quad (92)$$

It is easily observed that it suffices to check (92) for the generators, and that prime ones are enough. Moreover, the inequality is trivial for special alternating generators, so consider only the non-special ones. We tested that all such generators up to genus 4 satisfy (92). (For  $g = 2$ , and also  $n = 0$ , this property is implied by Ozsváth–Szabó's inequality (88).) Actually, very few generators have  $n < g(K)$ : only 4 of the non-special prime generators of genus 3 and 30 of those of genus 4 have coefficients showing a true “trapez”, rather than a “triangle”. We may mention that we did not test any analog of (92) for links.

**Remark 9.4** A minor addition to part 1 of the log-concavity conjecture 9.2 is that if we have equality in (85), then all 3 of  $X_{[k]}$ ,  $X_{[k\pm 1]}$  are equal. Our test verified this extra property, too, for knots up to genus 4 and the links in theorem 9.6.

### 9.3 Complete linear relations by degree

Let us remark that our proof of the Fox conjecture can be conceptually extended. While it is clear that one can verify a particular sort of linear inequality between coefficients of  $\Delta$  for given genus, we can in fact determine the *complete list* of such inequalities.

**Definition 9.1** A (convex) *polytope*  $\Sigma$  is the convex hull of a finite number of points in  $\mathbb{R}^n$  and rays  $\mathbb{R}_+ \cdot \vec{b}$  for  $\vec{b} \in \mathbb{R}^n$ . Alternatively, one may describe a polytope by a set of linear inequalities (or as intersection of half-spaces of  $\mathbb{R}^n$ ). If no rays occur, we call  $\Sigma$  *bounded*. If at most one point is there (with no point meaning that the vertex is the origin), then  $\Sigma$  is a *cone*. A *facet* of  $\Sigma$  is a codimension-1 subset of the boundary  $\partial\Sigma$  of  $\Sigma$ , given by intersection of  $\Sigma$  with a hyperplane.

There seems no uniform rule for the usage of ‘polytope’ and ‘polyhedron’ in the literature. Our attitude here is to avoid latter term throughout. The word ‘convex’ will be usually omitted since all polytopes we will deal with are such. Also note that, in contrast to us, many authors consider a polytope to be bounded per definition. Our terminology needs one more important clarification.

**Convention.** In the following the term *convex hull*  $\text{conv}(D)$  of a set  $D \subset \mathbb{R}^n$  is meant as *the closure* of the set

$$\left\{ \sum_{i=1}^m \lambda_i x_i : \sum_{i=1}^m \lambda_i = 1, \lambda_i \geq 0, x_i \in D \right\}.$$

The reason for taking closures is easily seen from the example of a line and a point (outside the line) in the plane. This phenomenon wildens in higher dimension: for a non-closed convex polytope only parts of some facets would be there. Formulated in terms of the linear inequalities describing the polytope, certain points for which some inequality is exact (i.e. an equality) are allowed, while others are not. The discussion of these cases puts no reasonable use-over-effort ratio in prospect. So we assume *all convex sets are closed* (and the inequalities describing the polytope are always non-strict).

We shall treat here genus 3 exemplarily. For technical reasons we formulate the result in terms of  $\nabla$ .

**Proposition 9.3** The signed coefficients  $\nabla_k = [\nabla]_k \cdot \text{sgn}([\nabla]_6)$  of  $\nabla(K)$  for an alternating knot  $K$  of genus 3 satisfy the following 6 inequalities:

$$\begin{aligned} \nabla_4 - \frac{\nabla_2 + 5\nabla_6}{2} &\leq \frac{1}{2}, & \nabla_4 - \nabla_2 - 3\nabla_6 &\leq 2, & \nabla_4 + 3\nabla_2 + 9\nabla_6 &\geq -4, \\ \nabla_4 + \frac{\nabla_2}{2} + 4\nabla_6 &\geq -\frac{1}{2}, & \nabla_4 - \nabla_2 + 7\nabla_6 &\geq -7, & \nabla_2 - 3\nabla_6 &\leq 4. \end{aligned} \quad (93)$$

This set of linear inequalities is complete up to constants, in the sense that any other linear inequality valid for all (or all but finitely many) alternating genus 3 knots is, up to a worse absolute term, a consequence of those above.

We will explain later how to remove the defect of the constants, but it requires a bit more sophisticated calculation.

**Proof.** If, for each generator  $D$ , we take all  $\tilde{\Delta}_i \neq \Delta(D)$ , then all have degree at most 4. (We will deal with the  $\Delta(D)$  later.) Let  $\nabla_i(z)$  be given by

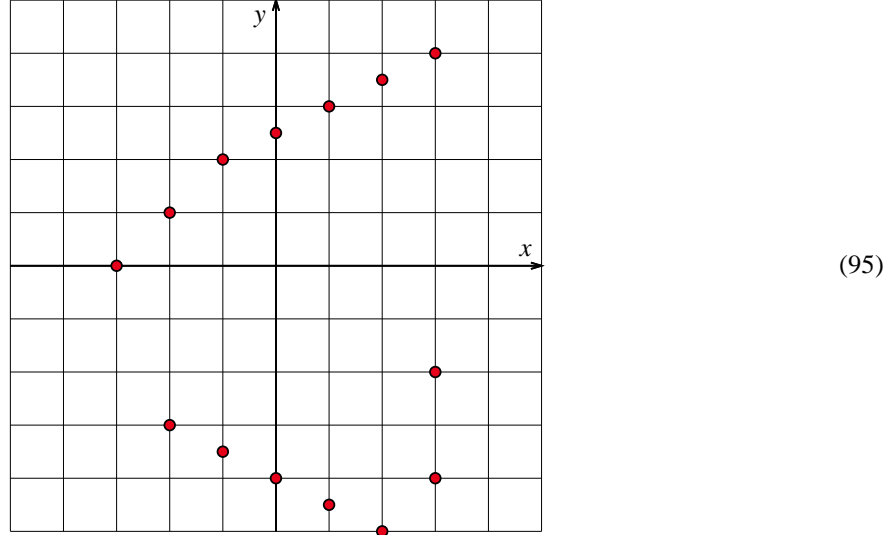
$$\nabla_i(t^{1/2} - t^{-1/2}) = \Delta_i(t) \cdot t^{-\max \deg \Delta_i/2}.$$

The reason for using  $\nabla_i(z)$  is that the factor  $1 - t$  in (89) turns into  $z$ , and so we have just a degree shift, which is faster on the computer. (Thus, in fact, we used  $\nabla_i$  instead of  $\Delta_i$  almost throughout the calculation for theorem 9.5.) Now  $\max \deg \tilde{\Delta}_i = \text{span } \nabla_i$  (recall §2.4).

Now we can do the previous sort of determination of extremal polynomials  $\nabla_i$  of span 4, but this time with all generators taken together. We collect again the corresponding (extremal) points in the plane

$$(x, y) = ([\nabla_i]_2 / [\nabla_i]_6, [\nabla_i]_4 / [\nabla_i]_6) \in \mathbb{R}^2, \quad (94)$$

as in (90), together with the two points (with  $x = 0$ ) describing the convex hull (which now is an interval) for the  $\nabla_i$  of span 0 and 2. Here is the result:



The convex hull  $\tilde{\Sigma}$  is a hexagon, described by equations (one for each boundary edge) in which the r.h.s. of an equation in (93) is set to 0,  $\nabla_2$  is replaced by  $x$ ,  $\nabla_4$  by  $y$ , and  $\nabla_6$  by 1. To obtain (93), we need then to calculate the minimal or maximal value of the l.h.s. over all generator polynomials  $\Delta(D)$  (which we previously omitted from the calculation).

To see completeness, note that for each  $\Delta_i \neq \Delta(D)$  one has a sequence of diagrams  $\bar{D}_j \in \langle D \rangle$  with

$$\lim_{j \rightarrow \infty} \frac{\Delta(\bar{D}_j)}{[\Delta(\bar{D}_j)]_0} = \frac{\Delta_i}{[\Delta_i]_0},$$

so the set of points (94) is not reducible.  $\square$

Let us write more generally  $\tilde{\Sigma}_g \subset \mathbb{R}_+^{g-1}$  for the convex sets obtained as in the preceding proof for generators of genus  $g$ . For  $g > 3$  the procedure remains largely the same. Note that these polytopes will always be bounded, due to proposition 9.1.

It is natural to restrict oneself also to knots of given signature, and figure 14 shows the pictures corresponding to (95) for  $g = 3$ . For example, for  $\sigma = 6$  (the special alternating knots), one derives the (complete up to constants linear) inequalities:

$$\nabla_4 - \frac{4}{3}\nabla_2 \geq -3, \quad \nabla_2 \geq 3, \quad \nabla_4 - \nabla_2 - 2\nabla_6 \leq -2, \quad \nabla_4 - \frac{\nabla_2 + 5\nabla_6}{2} \leq -\frac{1}{2}. \quad (96)$$

Let us denote the corresponding (bounded) polytopes as  $\tilde{\Sigma}_{g,\sigma}$ .

The graphics of figure 14 were displayed to show the difference to the properties considered unrelatedly to our approach. Roughly (i.e. ignoring the additive terms independent on  $\nabla_k$ ), the conditions (87) of Ozsváth–Szabó (only  $s = 0, 1$  are relevant here) would lead to the restrictions

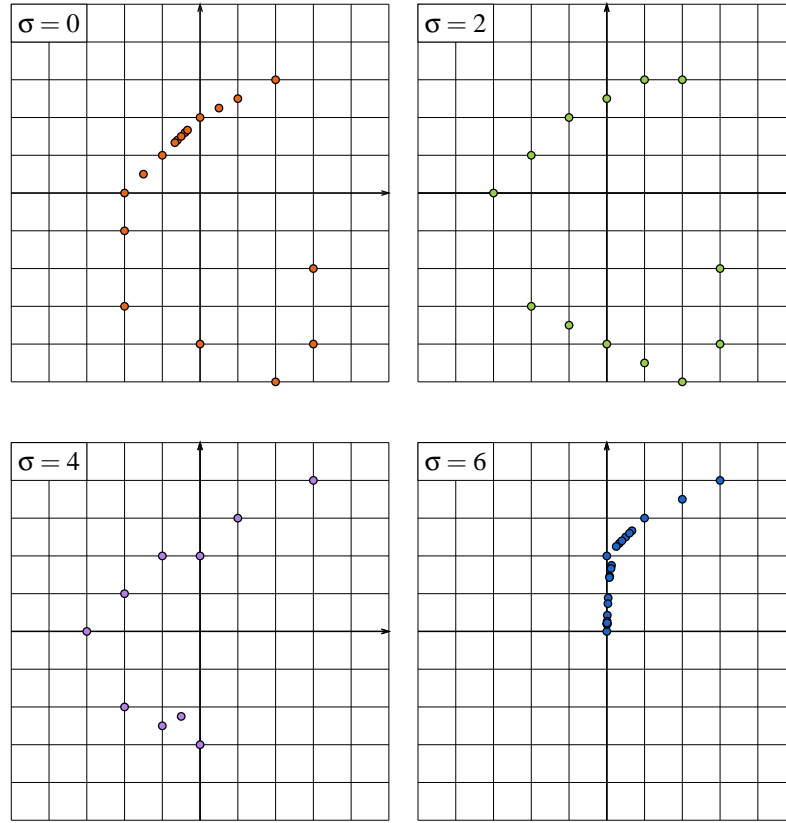
$$y \leq 4 \quad \text{and} \quad y \leq \frac{x+6}{2}, \quad (97)$$

while the trapezoidal inequalities (86) (with the options ‘>’ and ‘=’ merged into ‘ $\geq$ ’) transcribe into

$$y \leq \min \left( 5, \frac{x+5}{2}, \frac{x}{3} + 3 \right). \quad (98)$$

These inequalities are (as should be) easily seen to follow from the ones describing our polygons  $\tilde{\Sigma}$ , but as much weaker consequences. Note in particular that, even taken together, (97) and (98) restrict the possible  $(x, y)$  only





**Figure 14:** Extremal points of the polygons  $\tilde{\Sigma}$  for genus  $g = 3$  and given signature.

to an infinite region of the plane. Also, the dependence of the shape of  $\nabla$  on the signature becomes much more apparent in our polygons than in (87). This shows how much further information one can obtain with our method, if one considers a fixed genus. Still (86) or (87) may define (without abusing constants) a facet of  $\Sigma$ . We will test this explicitly in below for  $g = 3$ .

If one wants to remove the inaccuracy up to constants, we have the following statement, in general dimension, which is worth taking record of.

**Theorem 9.7** The (closed) convex hull  $\Sigma_g$  of the set of Conway polynomials of alternating knots of given genus  $g$ , or  $\Sigma_{g,\sigma}$  given genus and signature, is a convex polytope in  $\mathbb{R}^g$ .

**Proof.** To determine  $\Sigma_g$ , one must find the convex hull  $\tilde{\Sigma}_D$  of the points (94) for each generator  $D$  separately, take the cone  $\mathbb{R}_+ \cdot (\tilde{\Sigma}_D \times \{1\})$  of  $\tilde{\Sigma}_D$  in  $\mathbb{R}^g$ , and translate this cone to have the summit  $\nabla(D)$ . Then  $\Sigma_g$  is the convex hull of the union of all these translated cones taken over all generators  $D$ . This can be rewritten as

$$\Sigma_g = \mathbb{R}_+ \cdot (\tilde{\Sigma}_g \times \{1\}) + \text{conv} \left( \{ \nabla(D) : D \text{ is a generator of genus } g \} \right) \quad (99)$$

(with the '+' being the Minkowski set-theoretic sum  $A + B = \{a + b : a \in A, b \in B\}$ ). The result  $\Sigma_g$  is a(n unbounded) convex polytope, as follows from the Farkas-Minkowski-Weyl Theorem for convex polytopes (see e.g. [Sc], Corollary 7.1a).  $\square$

**Remark 9.5** Note that one can replace 'alternating' by 'positive' in theorem 9.7. (For knots of given genus and signature a bit extra argument is needed.) We chose not to elaborate on the positive case here, though.

Let us call a facet  $F \subset \partial\Sigma$  of  $\Sigma \subset \mathbb{R}^g$  to be *n-open* if it contains an affine-translated copy of  $\mathbb{R}_+^n \times \{0\}^{\times g-n}$ , but not one of  $\mathbb{R}_+^{n+1} \times \{0\}^{\times g-n-1}$ . Say  $F$  is *largest* if it is  $(g-1)$ -open. Then inequalities like the above (93) or (96), said to be complete up to constants, are in fact those for the largest facets of  $\Sigma_g$ .

Theorem 9.7 clearly also gives a practical way to determine  $\Sigma_g$ , though it requires to work in one dimension up in comparison to  $\tilde{\Sigma}_g$ . Latter would likely be quite more complicated to describe precisely for high genus. (In calculating  $\tilde{\Sigma}_3$  above, we took advantage of the 2-dimensional convex hull algorithm we had implemented for theorem 9.5.) Nevertheless this is feasible (even by hand) for genus 2, and carried out by In-Dae Jong [Jn2]. The result can be stated as follows.

**Theorem 9.8** The complete set of linear inequalities satisfied by the coefficients  $\nabla_i = [\nabla]_i \cdot \text{sgn}([\nabla]_4)$  of Conway polynomials of alternating genus 2 knots of given signature is:

$$\nabla_4 \geq 1 \text{ (for all } \sigma), \text{ and } \left\{ \begin{array}{ll} -2\nabla_4 - 1 \leq \nabla_2 \leq \nabla_4 + 1 & \text{for } \sigma = 0, \\ -2\nabla_4 + 1 \leq \nabla_2 \leq 2\nabla_4 - 1 & \text{for } |\sigma| = 2, \\ 2 \leq \nabla_2 \leq 2\nabla_4 + 1 & \text{for } |\sigma| = 4 \end{array} \right\}.$$

For genus 3 one must use a computer. There is software for converting a convex-hull (vertex-ray) representation of a polytope like (99) into a half-space intersection (linear inequalities) representation. We used the programs `cdd/cdd+` of Komei Fukuda [Fu] and `lrs` of David Avis [Av] to find a linear inequality representation for  $\Sigma_{3,\sigma}$  from (99) and the calculation of figure 14. (The programs do essentially the same, but use different algorithms, and we applied them both for consistency security.)

The description of  $\Sigma_{3,\sigma}$  is shown in the following table in columns 2 and 3. An entry of the form ‘ $mvr$ ’ in the third column means that  $\Sigma_{3,\sigma}$  is the convex hull of  $m$  vertices and  $n$  rays. (For example, ‘ $1v2r$ ’ is a planar angle segment, and ‘ $2v1r$ ’ is a plane half-strip.) The second column gives the number of facets of (or minimal number of linear inequalities describing)  $\Sigma_{3,\sigma}$ .

The following columns show the efficiency of the conditions (87) (with ‘ $OSn$ ’ standing for the case  $s = n-1$ ) and (86) (with ‘ $Tm$ ’ standing for the inequality obtained by putting  $k = m$  and joining the two alternatives into a non-strict inequality, without regard to the other  $k$ ) for  $g = 3$  and given signature  $\sigma$ . An entry of the form ‘ $mvr$ ’ is as in column 2 and describes here the intersection of  $\Sigma_{3,\sigma}$  with the hyperplane defined by the inequality being exact (i.e. an equality). In case the intersection is empty (the inequality is never exact), a bracketed number ‘ $[n]$ ’ means that the smallest defect of the absolute term is  $n$ . (That is, in an inequality  $x \leq y$  with  $x \in \mathcal{L}in\{\nabla_i\}$  and  $y \in \mathbb{R}$  the largest value of the l.h.s. on  $\Sigma_{3,\sigma}$  is  $y - n$ .) One sees that both the trapezoidal and Ozsváth–Szabó inequalities are (not more and not less than) moderately good as conditions on the Alexander polynomial for genus 3, and their sharpness increases with increasing (absolute) signature.

$\sigma \backslash \text{ineq.}$	f	vr	OS1	OS2	T1	T2	T3
0	10	5v7r	[2]	[1]	[2]	[2]	[2]
2	12	6v8r	[1]	1v	[1]	[1]	2v2r
4	8	4v5r	[1]	2v1r	[2]	1v1r	2v2r
6	6	3v4r	2v	1v1r	1v	1v1r	2v2r
total	12	6v6r	–	–	1v	2v1r	–

The determination of  $\Sigma_{3,\sigma}$  also leads to a description of  $\Sigma_3$ , stated in the below theorem. Some data is given also in the bottom line of the table. (Note that (87) depend, at least in the constant terms, slightly on  $\sigma$ , and so does T3 on  $\sigma \bmod 4$  when we sign  $\nabla_i$  so that  $\nabla_6 > 0$ . Thus we cannot compare  $\Sigma_3$  to these inequalities directly.)

**Theorem 9.9** The complete linear inequalities satisfied by the signed Conway coefficients  $\nabla_k = [\nabla]_k \cdot \text{sgn}([\nabla]_6)$  of an alternating knot of genus 3 are those in (93), together with the following 6 other inequalities:

$$\begin{aligned} 40\nabla_6 - \nabla_4 - 12\nabla_2 &\geq -37, & 34\nabla_6 + 8\nabla_4 + 3\nabla_2 &\geq -1, & 5\nabla_6 + \nabla_4 + \nabla_2 &\geq 0, \\ 3\nabla_6 + \nabla_2 &\geq -1, & 9\nabla_6 - 3\nabla_4 + \nabla_2 &\geq 0, & \nabla_6 &\geq 1. \end{aligned} \quad \square$$

We conclude with the table for  $g = 4$ .

$\sigma \backslash \text{ineq.}$	f	vr	OS1	OS2	OS3	T1	T2	T3	T4
0	105	26v26r	1v	[2]	1v	[1]	[1]	[1]	[1]
2	115	29v31r	[1]	[2]	[1]	[2]	[2]	[2]	4v4r
4	81	21v24r	1v	[1]	1v	[1]	[1]	2v2r	5v5r
6	41	11v14r	2v1r	[1]	2v1r	[2]	1v1r	2v2r	5v5r
8	18	5v8r	1v1r	3v	1v1r	1v	1v1r	2v2r	2v4r
total	100	27v26r	—	—	—	1v	2v1r	4v2r	—

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